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LIMITS OF EQUISYMMETRIC 1-COMPLEX DIMENSIONAL FAMILIES OF RIEMANN SURFACES

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Abstract

We describe the limit surfaces of some equisymmetric 1-complex dimensional families of Riemann surfaces in the boundary of the Deligne-Mumford compactification of moduli space. We provide a description of such nodal Riemann surfaces in terms of the group of automorphisms defining the family. We apply our method to some known examples.

1. Introduction

Let $\mathscr{M}_g$ be the moduli space of smooth curves of genus g and let $\widehat{\mathscr{M}_g}$ be the set of stable curves of genus g. A well-known result of Deligne and Mumford states that the set $\widehat{\mathscr{M}_g}$ can be endowed with a structure of projective complex variety and contains $\mathscr{M}_g$ as a dense open subvariety [10]. If $g \geq 2$ then $\widehat{\mathscr{M}_g}$ is an irreducible complex projective variety of dimension $3g - 3$.

The stable curves can be studied from the point of view of Bers as Riemann surfaces with nodes and as degeneration of Riemann surfaces (see [1], [2], [3], [13] and [16]). Thus an element of $\widehat{\mathscr{M}_g}$ can be considered as a stable curve or as a Riemann surface with nodes.

To each stable curve (or Riemann surface with nodes), of genus greater than 2, one can associate a connected weighted graph such that every vertex with weight 0 has at least three edges coming into it (if the graph is just a vertex the weight is always ≥ 2, the genus of the curve). This graph is called a stable graph. There is a bijection between the set of topological types of stable curves and the set of isomorphy classes of stable graphs.

Let $[C] \in \mathscr{M}_g$ and assume that $\pi\colon C \to \mathbb{P}^1(\mathbb{C})$ is a Galois branched covering of $\mathbb{P}^1(\mathbb{C})$ with branch locus $\{q_1, q_2, q_3, q_4\}$ such that the stabilizers subgroups of the points p of C on each q_i, $1 \leq i \leq 3$, are of order 2 and the stabilizer subgroups of the points in C on q_4 have order m with $m \geq 3$. We will denote by

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$\mathscr{F} \subset \mathscr{M}_g$ the 1-dimensional family of curves that admit a covering of $\mathbb{P}^1(\mathbb{C})$ topologically equivalent to π. There exist many such families, for example in [15] a description is given in terms of Fuchsian groups of these families for $3 \leq g \leq 10$ and $|G| \geq 4(g-1)$. The points in $\overline{\mathscr{F}} \cap (\widehat{\mathscr{M}_g} - \mathscr{M}_g) \neq \varnothing$ are non-smooth surfaces with nodes which arise as degeneration of surfaces of the family $\mathscr{F}$. The main goal of this article is to describe these "limit" points $L \in \overline{\mathscr{F}} \cap (\widehat{\mathscr{M}_g} - \mathscr{M}_g)$. We shall obtain not only the stable graph of every limit L but also its conformal structure.

In the second section we recall some well-known concepts and fix some notations, in the third we prove the main theorem which allows to describe the "limit" points, in the fourth we give some examples of the "limits" of some families and in some cases their description as families of projective curves.

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2. Preliminaries and notation

2.1. Equisymmetric families

We will denote by $\mathbb{D} = \{z \in \mathbb{C} : \|z\| < 1\}$ the unit disc. If $\Lambda \subset \mathrm{Aut}^+(\mathbb{D})$ is a cocompact Fuchsian group (discrete and cocompact subgroup of $\mathrm{Aut}^+(\mathbb{D})$) its signature is a collection of non-negative integers $\gamma, m_1 \geq 2, \ldots, m_k \geq 2$ and will be denoted by

$$s = (\gamma; [m_1, \ldots, m_k]). \tag{1}$$

The compact orbit space has the structure of an orientable 2-orbifold of topological genus γ with k conic points of orders $m_1, \ldots, m_k$. In the non-cocompact case the signature is $(\gamma; [m_1, \ldots, m_k]; t, r)$, where t is the number of cusps and r is the number of open boundary components.

Let $\mathfrak{G}$ be an abstract group isomorphic to a Fuchsian group with signature s as in (1). We denote by $R(s)$ the set of monomorphisms (faithful representations) $r\colon \mathfrak{G} \to \mathrm{Aut}^+(\mathbb{D})$ such that $r(\mathfrak{G})$ is a Fuchsian group with signature s. The set $R(s)$ has a natural topology given by the topology of $\mathrm{Aut}^+(\mathbb{D})$. Two elements r_1 and r_2 in $R(s)$ are said to be equivalent ($\sim$), if there exists $h \in \mathrm{Aut}^+(\mathbb{D})$ such that for each $a \in \mathfrak{G}$, $r_1(a) = h \circ r_2(a) \circ h^{-1}$. The topological space of classes $\mathbf{T}(s) = R(s)/\sim$ is the Teichmüller space of Fuchsian groups with signature s (see [14]).

The Teichmüller space $\mathbf{T}(s)$ is a complex manifold homeomorphic to $\mathbb{R}^{d(s)}$, where

$$d(s) = 3(2\gamma - 2) + 2k \tag{2}$$

and s is the signature given in (1).

Let G be a finite group generated by 3 elements a_1, a_2 and a_3 of order 2 and assume that $m \geq 3$ is the order of $a_1a_2a_3$. Let $\mathfrak{D}$ be an abstract group with presentation

$$\mathfrak{D} = \langle x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^2 = x_4^m = x_1x_2x_3x_4 = 1\rangle. \tag{3}$$

The existence of an epimorphism $\theta\colon \mathfrak{D} \to G$ is equivalent to having a presentation of G of the form

$$\langle a_1, a_2, a_3, a_4 : a_1^2 = a_2^2 = a_3^2 = a_4^m = a_1a_2a_3a_4 = 1, \ldots\rangle.$$

We have the exact sequence of groups

$$\{1\} \longrightarrow \mathfrak{K} \xrightarrow{i} \mathfrak{D} \xrightarrow{\theta} G \xrightarrow{p} \{1\},$$

where $\mathfrak{K} \cong \ker(\theta)$.

Let $r\colon \mathfrak{D} \to \mathrm{Aut}^+(\mathbb{D})$ be a monomorphism such that $[r] \in \mathbf{T}(s)$, where $s = (0; [2, 2, 2, m])$. We have a commutative diagram:

$$\begin{tikzcd}
\mathfrak{K} \arrow[r, "i"] \arrow[d, "r"'] & \mathfrak{D} \arrow[d, "r"] \\
r(\mathfrak{K}) \arrow[r, "\psi"'] & r(\mathfrak{D})
\end{tikzcd}$$

The Fuchsian group $r(\mathfrak{K})$ is a surface Fuchsian group (i.e. a Fuchsian group isomorphic to the fundamental group of a surface) which uniformize a compact connected Riemann surface $X \cong \mathbb{D}/r(\mathfrak{K})$ of genus

$$g = \frac{(m-2)\,|G|}{4m} + 1. \tag{4}$$

The natural projection $\pi\colon \mathbb{D}/r(\mathfrak{K}) \to \mathbb{D}/r(\mathfrak{D})$ is a regular branched covering with deck transformation group isomorphic to G. Furthermore the orbifold universal covering of such a 1-complex dimensional family of Riemann surfaces is $\mathbf{T}(s)$ which by (2) is homeomorphic to $\mathbb{R}^2$. If we denote by $\mathbf{T}_g$ the Teichmüller space of surfaces of genus g, the monomorphism $i\colon \mathfrak{K} = \ker(\theta) \hookrightarrow \mathfrak{D}$ induces an inclusion $i_*(\theta)\colon \mathbf{T}(s) \hookrightarrow \mathbf{T}_g$. The action of the modular group produces a commutative diagram:

$$\begin{tikzcd}
\mathbf{T}(s) \arrow[r, "i_*(\theta)"] \arrow[d, "p"'] & \mathbf{T}_g \arrow[d, "p"] \\
\mathcal{M}_g(G, \theta; s) \arrow[r] & \mathcal{M}_g
\end{tikzcd}$$

Let $\theta\colon \mathfrak{D} \to G$ be a fixed epimorphism. For any automorphism $\alpha\colon G \to G$ and any automorphism $\delta\colon \mathfrak{D} \to \mathfrak{D}$, we may construct a new epimorphism $\alpha\circ\theta\circ\delta\colon \mathfrak{D} \to G$. It can be remarked that the elements $\alpha\circ\theta\circ\delta(x_i), i = 1, \dots, 4$ give a new presentation of G. Note that $p \circ i_*(\theta) = p \circ i_*(\alpha \circ \theta \circ \delta)$.

Definition of an equisymmetric family:

DEFINITION 2.1. For an epimorphism $\theta\colon \mathfrak{D} \to G$ the set

$$\mathcal{M}_g(G, \theta; s) = p \circ i_*(\theta)(\mathbf{T}(s)) \subset \mathcal{M}_g$$

is called the *equisymmetric family* associated to $\theta\colon \mathfrak{D} \to G$.

2.2. Riemann surfaces with nodes

A connected complex analytic space S is a *Riemann surface with nodes* if and only if:

(1) there are $k = k(S) \geq 0$ points $p_1, \dots, p_k \in S$, called *nodes*, such that every node p_j has a neighbourhood isomorphic to the analytic set $\{z_1 z_2 = 0 : \|z_1\| < 1, \ \|z_2\| < 1\} \subset \mathbb{C}^2$ with p_j corresponding to $(0, 0)$,

(2) the set $S - \{p_1, \dots, p_k\}$ has $r \geq 1$ connected components $\Sigma_1, \dots, \Sigma_r$, called *components* of S, each of them is a Riemann surface of genus g_i, with n_i punctures with $3g_i - 3 + n_i \geq 0$ and $n_1 + \cdots + n_r = 2k$.

We denote by $g = (g_1 - 1) + \cdots + (g_r - 1) + k + 1$, the *genus* of S.

If $k = k(S) = 0$, S is called *non-singular* and if $k = k(S) = 3g - 3$, S is called *terminal*.

To a Riemann surface with nodes S we can associate a weighted graph, *the graph of* S, $\mathcal{G}(S) = (V_S, E_S, w)$, where V_S is the set of vertices, E_S is the set of edges, and w is a function from the set V_S and with non-negative integer values. This triple is defined in the following way.

(1) To each component Σ_i corresponds a vertex in V_S.

(2) To each node joining the components Σ_i and Σ_j corresponds an edge in E_S connecting the corresponding vertices. Multiple edges between the same pair of vertices and loops are allowed in $\mathcal{G}(S)$.

(3) The function $w\colon V_S \to \mathbb{Z}_{\geq 0}$ associates to any vertex the genus g_i of Σ_i.

Two surfaces with nodes S_1 and S_2 are homeomorphic if and only if $\mathcal{G}(S_1)$ and $\mathcal{G}(S_2)$ are isomorphic [17].

The group of automorphisms $\mathrm{Aut}(S)$ of S, a Riemann surface with nodes and of genus $g \geq 2$, is a finite group. We will denote by $\mathrm{Aut}(\mathcal{G}(S))$ the group of automorphisms of the graph. There is a homomorphism $\rho\colon \mathrm{Aut}(S) \to$

$\mathrm{Aut}(\mathcal{G}(S))$ which associates to any automorphism of a Riemann surface with nodes an automorphism of the associated graph. The homomorphism ρ may be neither injective nor surjective. Given a Riemann surface with nodes S and $\mathcal{G}(X)$ its associated stable graph, an automorphism $\widehat{\varphi} \in \mathrm{Aut}(\mathcal{G}(S))$ is said to be *geometric* if there exists $\varphi \in \mathrm{Aut}(S)$ such that $\rho(\varphi) = \widehat{\varphi}$.

3. The main theorem

Let G be a finite group generated by 3 elements a_1, a_2 and a_3 of order 2 in such a way that $a_1a_2a_3$ has order $m \geq 3$ or equivalently assume there is an epimorphism $\theta\colon \mathfrak{D} \to G$, where $\mathfrak{D}$ is a abstract group with presentation (3).

In this section we shall describe the surfaces with nodes which are limits of equisymmetric 1-complex dimensional families defined by $\theta\colon \mathfrak{D} \to G$.

3.1. The associated graph to the epimorphism θ

We begin by defining a weighted graph $\mathcal{G}(\theta)$ associated to $\theta\colon \mathfrak{D} \to G$. This graph will be the graph of surfaces with nodes which are limits of the uniparametric families.

Let $H(\theta)$ be the subgroup of G (possibly G) generated by $\theta(x_1x_2)$, $\theta(x_3)$ and $\theta(x_4)$. The vertices of $\mathcal{G}(\theta)$ are the right cosets of $H(\theta)$ in G; we write $H(\theta)\backslash G = \{H(\theta)h_i : i = 1, \ldots, r\}$.

Let $K(\theta)$ be the subgroup of $H(\theta)$ generated by $\theta(x_1x_2)$ and let $R = \{k_j : j = 1, \ldots, \ell\}$ be a set of representatives for the left cosets $G/K(\theta) = \{k_jK(\theta) : j = 1, \ldots, \ell\}$.

For each unordered pair $[k_i, k_j]$ of elements of R ($k_i \neq k_j$) such that

$$\theta(x_1)k_i \in k_jK(\theta),$$

there is an edge $e_{[k_i,k_j]}$ in $\mathcal{G}(\theta)$ joining the vertex with label $H(\theta)h_i$, where $k_i \in H(\theta)h_i$, with the labelled $H(\theta)h_j$ vertex, where $k_j \in H(\theta)h_j$. (If $H(\theta)h_i = H(\theta)h_j$, then $e_{[k_1,k_2]}$ is a loop.)

The weight of each vertex is

$$\frac{1}{2}\left(\frac{1}{2} - \frac{1}{m} - \frac{1}{|K(\theta)|}\right)|H(\theta)| + 1. \tag{5}$$

If $\alpha\colon G \to G$ is an isomorphism, then the graphs $\mathcal{G}(\theta)$ and $\mathcal{G}(\alpha \circ \theta)$ are isomorphic.

If $H(\theta) = G$, the associated graph $\mathcal{G}(\theta)$ has only 1 vertex, all the edges are self loops and we have the equality

$$\frac{1}{2}\left(\frac{1}{2} - \frac{1}{m} - \frac{1}{|K(\theta)|}\right)|H(\theta)| + \frac{|K(\theta)|}{2} = \frac{(m-2)\,|G|}{4m}.$$

THEOREM 3.1. *Assume $g \geq 2$. Let $\mathscr{M}_g(G, \theta; s)$ be the 1-complex dimensional equisymmetric family defined by $\theta\colon \mathfrak{D} \to G$. A noded Riemann surface L is in $\overline{\mathscr{M}_g(G, \theta; s)} \cap (\widehat{\mathscr{M}_g} - \mathscr{M}_g)$ if and only if there exists an automorphism $\delta \in \mathrm{Aut}^+(\mathfrak{D})$ such that:*

- *The graph of L is $\mathscr{G}(\theta \circ \delta)$. In particular, L has $[G : H(\theta \circ \delta)]$ components, where $[G : H(\theta \circ \delta)]$ denotes the index of $H(\theta \circ \delta)$ in G.*
- *Let $n = |\langle \theta(\delta(x_1 x_2)) \rangle|$, where $\langle \theta(\delta(x_1 x_2)) \rangle$ denotes the subgroup generated by $\theta(\delta(x_1 x_2))$ and let Ω be the triangular (spherical, Euclidean or hyperbolic) crystallographic group with signature $(0; [n, 2, m])$ and presentation*

$$\left\langle y_1, y_2, y_3 : y_1^n = y_2^2 = y_3^m = 1, y_1 y_2 y_3 = 1 \right\rangle .$$

If we define $\varphi\colon \Omega \to H(\theta \circ \delta)$ by

$$y_1 \mapsto \theta(\delta(x_1 x_2)), \quad y_2 \mapsto \theta(\delta(x_3)), \quad y_3 \mapsto \theta(\delta(x_4)),$$

then each component L_i of L is isomorphic to

$$(\mathbb{U} - \mathrm{Fix}\{y_1^z : z \in \Omega\}) / \ker \varphi,$$

where $\mathbb{U}$ is either $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$. Hence each component of L is a Platonic surface, where we pinch the preimages of a critical value with branch index n of $\mathbb{U} \to \mathbb{U}/\ker \varphi$.

PROOF. All Riemann surfaces of genus $g \geq 2$ that we consider will be endowed with a hyperbolic metric.

Let $L \in \overline{\mathscr{M}_g(G, \theta; s)} \cap (\widehat{\mathscr{M}_g} - \mathscr{M}_g)$, since $\mathscr{M}_g(G, \theta; s)$ is topologically a real surface, then there is a curve $t \longmapsto S_t$, where $S_t \in \mathscr{M}_g(G, \theta; s)$ and such that $S_t \to L$. We use *multigeodesic* for a finite set of disjoint simple closed geodesics in a hyperbolic surface. In S_t there is a multigeodesic $\{\gamma_t\} \subset S_t$ that collapses when we approach L. The fact that $S_t \in \mathscr{M}_g(G, \theta; s)$ tells us some properties of $\{\gamma_t\}$ related to the action of the automorphism group and allows us to describe the convergence $S_t \to L$ as we shall see later.

Let $G_t \subset Aut(S_t)$ be the corresponding automorphism groups isomorphic to G (for a generic t) given by the fact that $S_t \in \mathscr{M}_g(G, \theta; s)$. For each t, the quotient space S_t/G_t is a hyperbolic 2-orbifold of genus 0 with four conic points, three of them $p_1(t)$, $p_2(t)$, $p_3(t)$ of order two and one $p_4(t)$ of order m.

Since the lengths of the connected components of $\{\gamma_t\}$ tend to zero when $S_t \to L$, $\{\gamma_t\}$ is invariant by the action of G_t when S_t is near L; as we want to study L, we may assume that this is the case for all S_t. Hence $\{\gamma_t\}/G_t$ must be a set of simple and closed geodesics and/or simple geodesics arcs in the hyperbolic orbifold S_t/G_t.

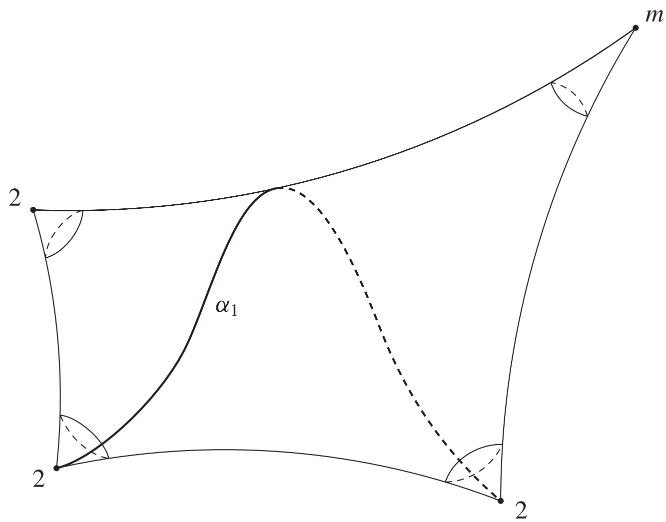


FIGURE 1

By the hyperbolic structure of the orbifolds S_t/G_t there are no simple closed geodesics in S_t/G_t (such a geodesic should split the orbifold in two pieces and one of them should not admit hyperbolic structure). Thus the possible orbit sets $\{\gamma_t\}/G_t$ are geodesic arcs joining a conic point of order two to another conic point of order two. Since the geodesics arcs $\{\gamma_t\}/G_t$ are disjoint, the unique possibility is that $\{\gamma_t\}/G_t$ is just one simple geodesic arc α_1 from a conic point $p_{\eta(1)}(t)$ of S_t/G_t to other one $p_{\eta(2)}(t)$, where $\eta(1), \eta(2) \in \{1, 2, 3\}$ (see Figure 1).

Note that given a simple arc a_{ij} from any conic point $p_i(t)$ of S_t/G_t to other conic point $p_j(t)$ there is a unique simple geodesic arc homotopically equivalent to a_{ij} and with the same ends than a_{ij} (see Theorems 1.6.6 and 1.6.7 of [5] or [11]). We denote $\eta(3) \in \{1, 2, 3\}$, in such a way that $\{\eta(1), \eta(2), \eta(3)\} = \{1, 2, 3\}$ and $\eta(4) = 4$. We can construct disjoint simple geodesic arcs α_i joining the conic points $p_{\eta(i)}(t)$, $p_{\eta(i+1)}(t)$ of S_t/G_t, $i = 1, 2, 3$, and such that α_1 is the arc $\{\gamma_t\}/G_t$.

Assume that $\pi_t\colon \mathbb{D} \to S_t/G_t$ is the universal covering of the hyperbolic orbifold S_t/G_t, with deck transformations group $r_t(\mathfrak{D})$. The group $r_t(\mathfrak{D})$ is a Fuchsian group with signature $(0; [2, 2, 2, m])$ (isomorphic to the fundamental orbifold group of S_t/G_t). The polygonal arc $\alpha_1 \cup \alpha_2 \cup \alpha_3$ lifts to the 1-skeleton $\pi_t^{-1}(\alpha_1 \cup \alpha_2 \cup \alpha_3)$ of a tessellation by pentagons of the complex disc and each pentagon is a fundamental region of the group $r_t(\mathfrak{D})$, thus each tessellation (in fact, each pentagon) determines a group $r_t(\mathfrak{D})$, with a canonical presentation, an orbifold S_t/G_t and, using the monodromy θ, a surface S_t. The curve $t \to S_t$

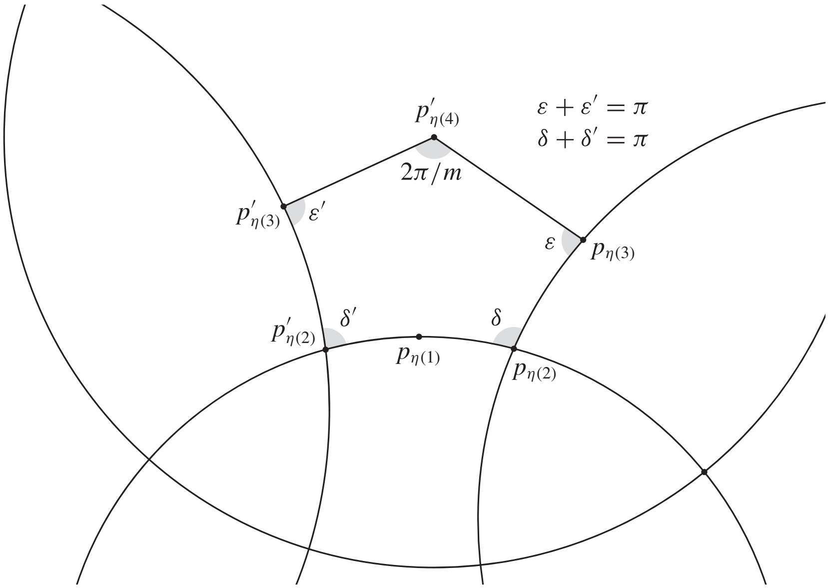


FIGURE 2

produces a parametrization of uniformization groups $t \to r_t(\mathfrak{D})$, these groups are determined by the tessellation of pentagons given by $\pi_t^{-1}(\alpha_1 \cup \alpha_2 \cup \alpha_3)$. The pentagonal tessellations degenerate to a tessellation by quadrilaterals with a vertex at infinity when the lengths of the sides $\pi_t^{-1}(\alpha_1)$ approach 0 (this corresponds to $S_t \to L$, i.e. the collapsing of the multigeodesic $\{\gamma_t\}$ in S_t). Any quadrilateral of the degenerated tessellation will be a fundamental region of an uniformization group of the quotient of the limit surface by the action of an automorphism group isomorphic to G and will provide us the information on L that we are looking for.

Let $\mathscr{P}_t$ be one of the pentagons of the tessellation with vertices $\widetilde{p}_{\eta(2)}(t)$, $\widetilde{p}_{\eta(3)}(t)$, $\widetilde{p}_{\eta(4)}(t)$, $\widetilde{p}'_{\eta(3)}(t)$ and $\widetilde{p}'_{\eta(2)}(t)$, such that $\pi_t(\widetilde{p}_{\eta(i)}(t)) = \pi_t(\widetilde{p}'_{\eta(i)}(t)) = p_{\eta(i)}(t)$. Let $\widetilde{p}_{\eta(1)}(t)$ be the middle point of the side $\widetilde{\alpha}_1$ of $\mathscr{P}_t$ in $\pi_t^{-1}(\alpha_1)$, note that $\pi_t(\widetilde{p}_{\eta(1)}(t)) = p_{\eta(1)}(t)$ (see Figure 2).

We let x'_j be the elliptic transformation with fixed point $\widetilde{p}_{\eta(j)}(t)$, $j = 1, 2, 3$, and multiplier $e^{\pi i}$, and let x'_4 be the elliptic transformation with fixed point $\widetilde{p}_4(t)$ and multiplier $e^{\frac{2\pi i}{m}}$. We have that

$$\left\langle x'_1, x'_2, x'_3, x'_4 : x'^2_1 = x'^2_2 = x'^2_3 = x'^m_4 = x'_1 x'_2 x'_3 x'_4 = 1 \right\rangle$$

is a presentation of $r_t(\mathfrak{D})$. If the canonical presentation of $r_t(\mathfrak{D})$ given by r_t

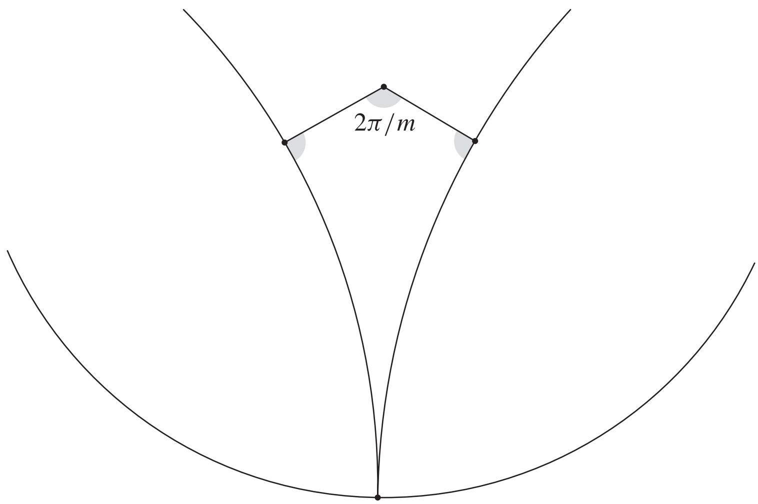


FIGURE 3

and the starting presentation of $\mathfrak{D}$ is

$$\langle x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^2 = x_4^m = x_1x_2x_3x_4 = 1\rangle,$$

then there is an automorphism β of $r_t(\mathfrak{D})$ such that $\beta(x_i) = x_i'$. The automorphism β produces the automorphism $\delta \in \mathrm{Aut}^+(\mathfrak{D})$ that we are looking for.

When S_t approaches L, the sides $\widetilde{\alpha}_1$ of the pentagons $\mathscr{P}_t$ approach infinity. The Fuchsian groups $r_t(\mathfrak{D})$ approach a non-cocompact Fuchsian group Δ with presentation

$$\Delta = \langle z, y_1, y_2 : y_1^2 = y_2^m = zy_1y_2 = 1\rangle,$$

where $x_1'x_2'$ approaches the parabolic transformation z, x_3' approaches y_1 and x_2' approaches y_2. The group Δ has signature $(0; [2, m]; 1; -)$ and has a fundamental region which is the quadrilateral shown in Figure 3, with a vertex at infinity.

Then each component of L is uniformized by the group $\ker \Psi \le \Delta$, where Ψ is defined by $\Psi(y_1) = \theta \circ \delta(x_3)$, $\Psi(y_2) = \theta \circ \delta(x_4)$ and $\Psi(z) = \theta \circ \delta(x_1x_2)$. The Riemann surface with cusps $\mathbb{D}/\ker \Psi$ can be completed to a compact Riemann surface $\widehat{\mathbb{D}/\ker \Psi}$ that is a covering of $\widehat{\mathbb{C}}$ branched at three points. Then $\widehat{\mathbb{D}/\ker \Psi}$ is uniformized by the surface subgroup $\ker\varphi$ of a triangular crystallographic group Ω of signature $(0; [n, 2, m])$, where n is the order of $\theta \circ \delta(x_1x_2)$ in the finite group G and φ is as defined in the statement of the theorem. Note that Ω acts on $\mathbb{U} = \widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$, depending on the value of $\frac{1}{n} + \frac{1}{2} + \frac{1}{m}$, and $\mathbb{U}/\ker\varphi$ is a Platonic surface (a Riemann surface underlying a regular map). Hence $\mathbb{D}/\ker\Psi$ is isomorphic to $(\mathbb{U} - \mathrm{Fix}\{y_1^z : z \in \Omega\})/\ker\varphi$ as claimed in the theorem.

The weight of each vertex in the graph of L, given in formula (5), is a consequence of the Riemann-Hurwitz formula applied to the covering $\mathbb{D}/\ker\Psi \to \mathbb{D}/\Omega$.

The lift of the geodesic polygonal $\alpha_1 \cup \alpha_2 \cup \alpha_3$ in S_t/G_t to S_t produces a tessellation by pentagons of S_t that is equivariant under the action of G_t. Suppose we label one of the pentagons $\mathscr{P}$ of the tessellation by $1 \in G, \mathscr{P} = \mathscr{P}_1$. Then defining $\mathscr{P}_g := g(\mathscr{P}_1)$, we obtain a complete labelling of the pentagons $\{\mathscr{P}_g\}$ of the tessellation of S_t.

If we now cut S_t by the multigeodesic $\{\gamma_t\}$, we obtain $\#G/H(\theta\circ\delta)$ connected components, and in each component appears all the pentagons with label in a given coset of $H(\theta \circ \delta)\backslash G$. We label each connected component of $S_t - \{\gamma_t\}$ by the coset Hg_i of $H\backslash G$ such that the pentagons in such component have labels belonging to Hg_i.

Two components with labels Hg_i and Hg_k of $S_t - \{\gamma_t\}$ are adjacent along a component of $\{\gamma_t\}$ if there are two pentagons $\mathscr{P}_a$ and $\mathscr{P}_b$ in the tessellation of S_t such that $a \in Hg_i$, $b \in Hg_k$ and $\mathscr{P}_a$, $\mathscr{P}_b$ have a common side in the preimage of α_1 under $S_t \to S_t/G_t$, i.e. $a \in k_1\langle\theta \circ \delta(x_1x_2)\rangle$, $b \in k_2\langle\theta \circ \delta(x_1x_2)\rangle$ and $(\theta \circ \delta(x_1))k_1 = k_2$. Then in the graph of L, the vertices corresponding to Hg_i and Hg_k must be joined. Hence the graph of L is $\mathscr{G}(\theta \circ \delta)$, since two vertices are adjacent if they correspond to components of L that are joined by a node.

Now given $\delta \in \mathrm{Aut}^+(\mathfrak{D})$, such an automorphism can be realized as an automorphism ζ of the orbifold S/G, where S is a generic surface of the family $\mathscr{M}_g(G, \theta, s)$. Let ξ be a curve from p_1 to p_2 such that lifts by $\mathbb{D} \to S/G$ to an arc joining the fixed points of the elliptic transformations x_1 and x_2. The lift of $\zeta^{-1}(\xi)$ by $S \to S/G$, is the multigeodesic $\{\gamma\}$ that we must contract to obtain the limit point L. The theorem is proved.

REMARK 3.2.

1. If $G = H$, the graph $\mathscr{G}(\theta \circ \delta)$ has only one vertex, with $[G : \langle\theta \circ \delta(x_1x_2)\rangle]/2$ self loops.
2. The graph $\mathscr{G}(\theta\circ\delta)$ is regular. The group G acts on $\mathscr{G}(\theta\circ\delta)$, but may be not faithfully. The valency of each vertex of $\mathscr{G}(\theta\circ\delta)$ is $[H : \langle\theta\circ\delta(\alpha(x_1x_2))\rangle]$.
3. It is possible to do a similar study for families $\mathscr{M}_g(G, \theta; [p, q, r, s])$. See [6] for a special case.
4. The group Ω may be a spherical, Euclidean or hyperbolic crystallographic group.
5. The automorphisms δ considered in the theorem correspond to different

presentations of G of the form

$$\langle a_1, a_2, a_3, a_4 : a_1^2 = a_2^2 = a_3^2 = a_4^m = a_1a_2a_3a_4 = 1, \ldots\rangle.$$

3.2. How to obtain families $\mathscr{M}_g(G, \theta; s)$

Let M be a reflexive regular map on a surface of genus $g \geq 2$ of type (p, q, r). The group of orientation preserving and reversing automorphisms G is a finite group generated by three elements of order two s_1, s_2, s_3 such that $(s_1s_2)^p = (s_2s_3)^q = (s_3s_1)^r = 1$. There are infinitely many groups of this type. Now we can construct an equisymmetric 1-dimensional family using G. It is easy to show that $s_4^{-1} = s_1s_2s_3$ has order $m > 1$.

Consider the abstract group $\mathfrak{G} = \langle a_1, a_2, a_3, a_4 : a_i^2 = 1,\ i = 1, 2, 3,\ a_4^m = 1,\ a_1a_2a_3a_4 = 1\rangle$ and the epimorphism $\theta\colon \mathfrak{G} \to G$ defined by $\theta(a_i) = s_i$, $i = 1, 2, 3, 4$. Using θ we can define families $\mathscr{M}_g(G, \theta; s)$ as considered in the main theorem.

4. Examples and some projective realizations

In each example we will describe: the finite group G, the generators a_1, a_2, a_3 and a_4, and the corresponding 1-complex dimensional equisymmetric family $\mathscr{M}_g(G, \theta; s) \subset \mathscr{M}_g$. For each Riemann surface with nodes L in $\widehat{\mathscr{M}_g}(G, \theta; s) \cap (\widehat{\mathscr{M}_g} - \mathscr{M}_g)$ we determine: its number of components, the crystallographic subgroup which uniformize each component, the genus of the normalization of each component and the associated stable graph $\mathscr{G}(L)$.

In some of the examples we give a projective realization of the 1-complex dimensional families and an explicit realization of theirs "limits".

4.1. Examples

4.1.1. The group $\mathfrak{S}_4$ in genus 3. Let $G \cong \mathfrak{S}_4$ be the symmetric group of degree 4 with generators $a_1 = (12)(34)$, $a_2 = (34)$ and $a_3 = (13)$. Let $a_4 = a_1a_2a_3 = (123)$. We define the epimorphism $\theta\colon \mathfrak{D} \to \mathfrak{S}_4$ by $\theta(x_i) = a_i$, for $i = 1, 2, 3$, and $\theta(x_4) = a_4^{-1}$.

The 1-complex dimensional equisymmetric family is $\mathscr{M}_3(\mathfrak{S}_4, \theta; s) \subset \mathscr{M}_3$, where $s = (0; [2, 2, 2, 3])$.

(1) Let $\delta = \mathrm{Id} \in \mathrm{Aut}^+(\mathfrak{D})$ in Theorem 3.1, then

$$H(\theta) = \langle\theta(x_1x_2) = (12), \theta(x_3) = (13), \theta(x_4) = (132)\rangle.$$

Let $L \in \widehat{\mathscr{M}_3}(\mathfrak{S}_4, \theta; s) \cap (\widehat{\mathscr{M}_3} - \mathscr{M}_3)$. Since $H(\theta) \cong \mathfrak{S}_3$, it follows that L has 4 components. We have $\langle\theta(x_1x_2) = (12)\rangle \cong \mathbb{Z}_2$, $n = 2$, so each component L_i, $i = 1, \dots, 4$, of L, is uniformized by a crystallographic group of signature (0; [2, 2, 3]) (a spherical group) and the normalization of each component L_i has genus $g = 0$. The associated graph $\mathcal{G}(\theta)$ has 4 vertices, each one with valency $[\mathfrak{S}_3 : \mathbb{Z}_2] = 3$ and weight 0.

Since $[\mathfrak{S}_4 : \langle (12) \rangle] = 12$, we have that $\mathcal{G}(\theta)$ has 6 edges and from each vertex there are 3 edges to the 3 remaining vertices. The graph $\mathcal{G}(\theta)$ is isomorphic to the 1-skeleton of a *tetrahedron*.

(2) Let $\delta \in \mathrm{Aut}^+(\mathfrak{D})$ be given by

$$\delta(x_1) = x_2, \quad \delta(x_2) = x_3, \quad \delta(x_3) = (x_2x_3)^{-1}x_1x_2x_3 \quad \text{and} \quad \delta(x_4) = (x_1x_2x_3)^{-1},$$

then

$$H(\theta \circ \delta) = \big\langle \theta \circ \delta(x_1x_2) = (134), \theta \circ \delta(x_3) = (14)(23), \theta \circ \delta(x_4^{-1}) = (132) \big\rangle.$$

Let $L \in \widehat{\mathcal{M}_3}(\mathfrak{S}_4, \theta; s) \cap (\widehat{\mathcal{M}_3} - \mathcal{M}_3)$. Note that $H(\theta \circ \delta) \cong \mathfrak{A}_4$. The nodal surface L has 2 components. Since $\langle \theta \circ \delta(x_1x_2) = (134) \rangle \cong \mathbb{Z}_3$, $n = 3$, each component L_i, $i = 1, 2$, of L, is uniformized by a crystallographic group with signature (0; [3, 2, 3]) (the normalization of each component L_i has genus $g = 0$). The associated graph $\mathcal{G}(\theta \circ \delta)$ has 2 vertices, each one with valency $[\mathfrak{A}_4 : \mathbb{Z}_3] = 4$ and with weight 0.

Since $[\mathfrak{S}_4 : \langle (134) \rangle] = 8$, we have that $\mathcal{G}(\theta \circ \delta)$ has 4 edges, from each vertex there are 4 edges to the other vertex. The graph $\mathcal{G}(\theta \circ \delta)$ is the 4-dipole.

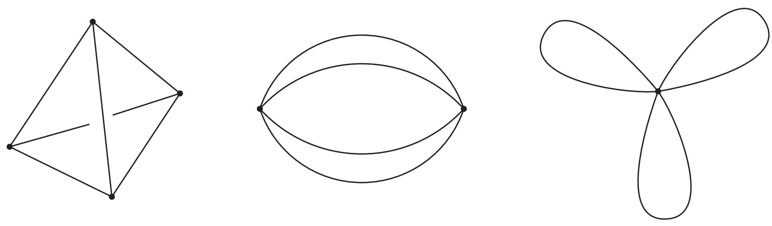
(3) Let $\beta \in \mathrm{Aut}^+(\mathfrak{D})$ be given by:

$$\beta(x_1) = x_1, \quad \beta(x_2) = x_3, \quad \beta(x_3) = (x_1x_3)^{-1}(x_1^{-1}x_2x_1)x_1x_3 \quad \text{and} \quad \beta(x_4) = (x_1x_2x_3)^{-1},$$

then

$$H(\theta \circ \beta) = \big\langle \theta \circ \beta(x_1x_2) = (1234), \theta \circ \beta(x_3) = (14), \theta \circ \beta(x_4^{-1}) = (132) \big\rangle.$$

Let $L \in \widehat{\mathcal{M}_3}(\mathfrak{S}_4, \theta; s) \cap (\widehat{\mathcal{M}_3} - \mathcal{M}_3)$. Now $H(\theta \circ \beta) \cong \mathfrak{S}_4$ and L has 1 component. Since $\langle (1234) \rangle = \mathbb{Z}_4$, $n = 4$, L is uniformized by a crystallographic group with signature (0; [4, 2, 3]) and the normalization of L has genus $g = 0$. Therefore L is a Riemann surface with nodes which has only one component. The graph $\mathcal{G}(\theta \circ \beta)$ has 1 vertex with



Id $\qquad$ δ $\qquad$ β

FIGURE 4

3 loops ($[\mathfrak{S}_4 : \langle(1234)\rangle] = 6$, L has 3 edges which correspond to the 3 loops). See Figure 4.

4.1.2. The group $\mathfrak{D}_p \times \mathfrak{D}_p$ in genus $g = (p-1)^2$. Let $p \geq 3$ be a prime number. We consider two copies of $\mathfrak{D}_p$, given by

$$\mathfrak{D}_p = \langle a, u : a^p = 1,\ u^2 = 1,\ uau = a^{p-1}\rangle,$$
$$\mathfrak{D}_p = \langle b, v : b^p = 1,\ v^2 = 1,\ vbv = b^{p-1}\rangle.$$

Let $G \cong \mathfrak{D}_p \times \mathfrak{D}_p$ and the generators $a_1 = u$, $a_2 = v$ and $a_3 = uvab$.

We recall that in this case the epimorphism $\theta\colon \mathfrak{D} \to \mathfrak{D}_p \times \mathfrak{D}_p$ is given by $\theta(x_1) = u$, $\theta(x_2) = v$, $\theta(x_3) = uvab$, and $\theta(x_4) = a^{(p-1)}b^{(p-1)}$. Furthermore the signature s is $(0; [2, 2, 2, p])$.

The 1-complex dimensional equisymmetric family is $\mathcal{M}_{(p-1)^2}(\mathfrak{D}_p \times \mathfrak{D}_p, \theta; s) \subset \mathcal{M}_{(p-1)^2}$.

(1) Let $\delta = \mathrm{Id} \in \mathrm{Aut}^+(\mathfrak{D})$ in Theorem 3.1. We have

$$H(\theta) = \langle \theta(x_1x_2) = uv,\ \theta(x_3) = uvab,\ \theta(x_4) = a^{(p-1)}b^{(p-1)}\rangle.$$

Let $L \in \widehat{\mathcal{M}_{(p-1)^2}}(\mathfrak{D}_p \times \mathfrak{D}_p, \theta; s) \cap (\widehat{\mathcal{M}_{(p-1)^2}} - \mathcal{M}_{(p-1)^2})$. Then $H(\theta) \cong \mathfrak{D}_p$ and L has $2p$ components. Since $\langle uv\rangle \cong \mathbb{Z}_2$, $n = 2$ and each component L_i, $i = 1, \ldots, 2p$, of L, is uniformized by a crystallographic group with signature $(0; [2, 2, p])$, then the normalization of each component L_i has genus $g = 0$. The associated graph $\mathcal{G}(\theta)$ has $2p$ vertices, each one with valency $[\mathfrak{D}_p : \mathbb{Z}_2] = p$ and weight 0. Since $[\mathfrak{D}_p \times \mathfrak{D}_p : \langle st\rangle] = 2p^2$, L has p^2 edges. Applying Theorem 3.1, we have that $\mathcal{G}(\theta)$ is the complete bipartite graph $K_{p,p}$.

(2) Let $\delta \in \mathrm{Aut}^+(\mathfrak{D})$ be given by: $\delta(x_1) = x_2$, $\delta(x_2) = x_3$, $\delta(x_3) = (x_2x_3)^{-1}x_1x_2x_3$ and $\delta(x_4) = (x_1x_2x_3)^{-1}$. Now we have

$$H(\theta\circ\delta) = \langle \theta\circ\delta(x_1x_2) = uab, \theta\circ\delta(x_3) = ua^2, \theta\circ\delta(x_4^{-1}) = a^{(p-1)}b^{(p-1)}\rangle.$$

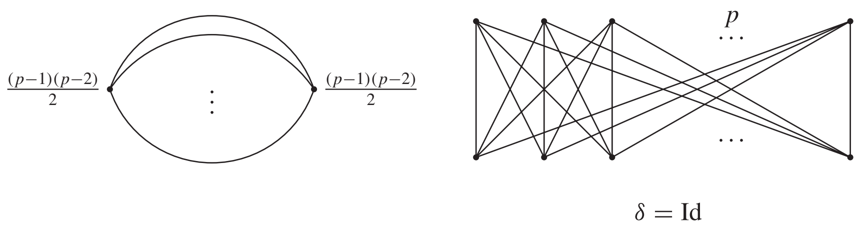


FIGURE 5

Let $c = uab$, $d = ua^2$ and $e = a^{(p-1)}b^{(p-1)}$, then $ce = u$, $c^2 = b^2$, $e^2c^2 = a^{-2}$, $\langle a, u\rangle \subset H(\theta \circ \delta)$ and $\langle b\rangle \subset H(\theta \circ \delta)$. Furthermore, v is not contained in $H(\theta \circ \delta)$, therefore $H(\theta \circ \delta)$ is isomorphic to a normal subgroup $\mathfrak{G}_{2p^2}$ of order $2p^2$ and L has 2 components.

Since $\langle uab\rangle \cong \mathbb{Z}_{2p}$, $n = 2p$ and each component L_i, $i = 1, 2$, of L, is uniformized by a crystallographic group with signature $(0; [2p, 2, p])$ and the normalization of each component L_i has genus $\frac{(p-1)(p-2)}{2}$. The associated graph $\mathcal{G}(\theta \circ \delta)$ has 2 vertices, each one with valency $[\mathfrak{G}_{2p^2} : \mathbb{Z}_{2p}] = p$ and with weight $\frac{(p-1)(p-2)}{2}$.

Since $[\mathfrak{D}_p \times \mathfrak{D}_p : \langle uab\rangle] = 2p$, $\mathcal{G}(\theta \circ \delta)$ is the p-dipole with p edges.

(3) Let $\beta \in \mathrm{Aut}^+(\mathfrak{D})$ be given by: $\beta(x_1) = x_1$, $\beta(x_2) = x_3$, $\beta(x_3) = (x_1x_3)^{-1}(x_1^{-1}x_2x_1)(x_1x_3)$ and $\beta(x_4) = (x_1x_2x_3)^{-1}$. Then

$$H(\theta \circ \beta) = \left\langle \theta \circ \beta(x_1x_2) = vab, \theta \circ \alpha(x_3) = vb^2, \theta \circ \beta(x_4^{-1}) = a^{(p-1)}b^{(p-1)} \right\rangle.$$

In this case we obtain a limit that is isomorphic to the one in the preceding case. See Figure 5.

REMARK 4.1. The 1-complex dimensional equisymmetric family $\mathcal{M}_{(p-1)^2}(\mathfrak{D}_p \times \mathfrak{D}_p, \theta; s) \subset \mathcal{M}_{(p-1)^2}$ was introduced in [7], [8], [9] and [19] as a 1-parameter family of cyclic p-gonal Riemann surfaces of genus $(p-1)^2$ admitting two p-gonal morphisms. Also in the particular case $p = 3$, in [4] a pencil of curves corresponding to the 1-complex dimensional equisymmetric family $\mathcal{M}_4(\mathfrak{D}_3 \times \mathfrak{D}_3, \theta; s) \subset \mathcal{M}_4$ was introduced. This pencil is related to a family of abelian surfaces with polarization of type $(1, 3)$.

In the previous example when $p = 3$, $\mathcal{M}_4(\mathfrak{D}_3 \times \mathfrak{D}_3, \theta; s) \cap (\widehat{\mathcal{M}_4} - \mathcal{M}_4)$ consists of the Riemann surfaces with nodes which have the following associated graphs:

(1) $\mathcal{G}(\theta)$ is isomorphic to the complete bipartite graph $K_{3,3}$,

(2) $\mathcal{G}(\theta \circ \delta)$ is a graph with 2 vertices with weight 1 connected by 3 edges.

4.1.3. The group $\mathfrak{A}_5$ *in genus* $g = 6$. Let $G \cong \mathfrak{A}_5$ be the alternating group of degree 5 and let $a_1 = (12)(45), a_2 = (12)(34), a_3 = (23)(45), a_4 = (235)$. We consider the epimorphism $\theta \colon \mathfrak{D} \to \mathfrak{A}_5$ determined by: $\theta(x_1) = a_1, \theta(x_2) = a_2$, $\theta(x_3) = a_3$ and $\theta(x_4) = a_4^{-1}$.

The 1-complex dimensional equisymmetric family is $\mathscr{M}_6(\mathfrak{A}_5, \theta; s) \subset \mathscr{M}_6$, where $s = (0; [2, 2, 2, 3])$.

(1) Let $\delta = \mathrm{Id} \in \mathrm{Aut}^+(\mathfrak{D})$ in Theorem 3.1. We have

$$H(\theta) = \big\langle \theta(x_1x_2) = (345),\ \theta(x_3) = (23)(45),\ \theta(x_4) = (253)\big\rangle.$$

Let $L \in \widehat{\mathscr{M}_6}(\mathfrak{A}_5, \theta; s) \cap (\widehat{\mathscr{M}_6} - \mathscr{M}_6)$, $H(\theta) \cong \mathfrak{A}_4$. The surface with nodes L has 5 components L_i, $i = 1, \dots, 5$, and the associated graph $\mathscr{G}(\theta)$ has 5 vertices.

Since $\langle (345) \rangle \cong \mathbb{Z}_3, n = 3$, then each component $L_i, i = 1, \dots, 5$, of L, is uniformized by a crystallographic group with signature (0; [3, 2, 3]) (the normalization of each component L_i has genus $g = 0$). The associated graph $\mathscr{G}(\theta)$ has 5 vertices, each one with valency $[\mathfrak{A}_4 : \mathbb{Z}_3] = 4$ and weight 0.

Since $[\mathfrak{A}_5 : \langle a_1a_2 \rangle] = 20$, the graph $\mathscr{G}(\theta)$ has 10 edges. From each vertex of $\mathscr{G}(\theta)$ there are 4 edges to the 4 remaining vertices. The graph $\mathscr{G}(\theta)$ is K_5, the complete graph on five vertices.

(2) Let $\delta \in \mathrm{Aut}^+(\mathfrak{D})$ in Theorem 3.1 be given by: $\delta(x_1) = x_2$, $\delta(x_2) = x_3$, $\delta(x_3) = (x_2x_3)^{-1}x_1x_2x_3$ and $\delta(x_4) = (x_1x_2x_3)^{-1}$. We have

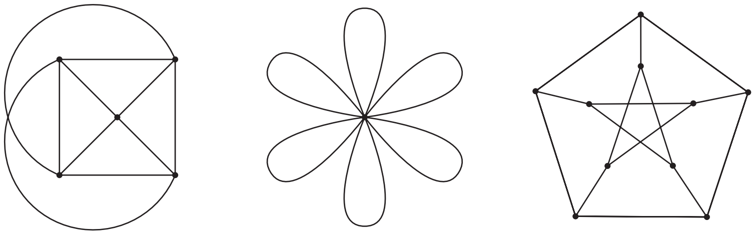
$$\begin{aligned} H(\theta \circ \delta) = \big\langle \theta \circ \delta(x_1x_2) = (13542),& \\ \theta \circ \delta(x_3) = (13)(24), \theta \circ \delta(x_4) = (253)\big\rangle& \end{aligned}$$

Let $L \in \widehat{\mathscr{M}_6}(\mathfrak{A}_5, \theta; s) \cap (\widehat{\mathscr{M}_6} - \mathscr{M}_6)$. Now $H(\theta \circ \delta) \cong \mathfrak{A}_5$ and L has 1 component. Since $\langle (13542) \rangle \cong \mathbb{Z}_5$, $n = 5$, L is uniformized by a crystallographic group with signature (0; [5, 2, 3]) and the normalization of L has genus $g = 0$. Therefore L is a Riemann surface with nodes which has only one component, since the genus of the normalization $\widehat{L}$ is 0, L has 6 nodes. Its graph $\mathscr{G}(\theta \circ \delta)$ has 1 vertex and 6 loops. Also applying Theorem 3.1 and since $[\mathfrak{A}_5 : \langle \theta \circ \delta(x_1x_2) \rangle] = 12$, we obtain again that $\mathscr{G}(\theta \circ \delta)$ has 6 loops.

Now we consider an epimorphism $\vartheta \colon \mathfrak{D} \to \mathfrak{A}_5$:

(3) Let $G \cong \mathfrak{A}_5$ and $b_1 = (23)(45)$, $b_2 = (24)(35)$, $b_3 = (12)(34)$, $b_4 = (125)$. Let $\mathfrak{D}$ be given as

$$\mathfrak{D} = \big\langle y_1, y_2, y_3, y_4 : y_1^2 = y_2^2 = y_3^2 = y_4^m = y_1y_2y_3y_4 = 1\big\rangle.$$



Id $\qquad$ δ $\qquad$ ϑ

FIGURE 6

We consider the epimorphism $\vartheta\colon \mathfrak{D} \to \mathfrak{A}_5$ determined by: $\vartheta(y_1) = b_1$, $\vartheta(y_2) = b_2$, $\vartheta(y_3) = b_3$ and $\vartheta(y_4) = b_4^{-1}$. The 1-complex dimensional equisymmetric family is $\mathcal{M}_6(\mathfrak{A}_5, \vartheta; s) \subset \mathcal{M}_6$, where $s = (0; [2, 2, 2, 3])$. Let $\delta = \mathrm{Id} \in \mathrm{Aut}^+(\mathfrak{D})$ in Theorem 3.1. We have

$$H(\vartheta) = \big\langle \vartheta(y_1 y_2) = (25)(34), \vartheta(y_3) = (12)(34), \vartheta(y_4) = (152)\big\rangle.$$

Let $L \in \widehat{\mathcal{M}_6}(\mathfrak{A}_5, \vartheta; s) \cap (\widehat{\mathcal{M}_6} - \mathcal{M}_6)$. We have $H(\vartheta) \cong \mathfrak{S}_3$ and L has 10 components L_i, $i = 1, \dots, 10$; the associated graph $\mathcal{G}(\vartheta)$ has 10 vertices. Since $\langle(25)(34)\rangle \cong \mathbb{Z}_2$, $n = 2$, each component L_i, $i = 1, \dots, 5$, of L, is uniformized by a crystallographic group with signature $(0; [2, 2, 3])$ and the normalization of each component L_i has genus $g = 0$. The associated graph $\mathcal{G}(\vartheta)$ has 10 vertices, each one with valency $[\mathfrak{S}_3 : \mathbb{Z}_2] = 3$ and with weight 0. Since $[\mathfrak{A}_5 : \langle(25)(34)\rangle] = 30$ then $\mathcal{G}(\vartheta)$ has 15 edges. By applying the description of the graph before Theorem 3.1 we obtain that $\mathcal{G}(\vartheta)$ is the Petersen graph. See Figure 6.

4.2. Some projective realizations

The algebraic variety structure for the set $\mathcal{M}_g$ is given in such a way that for any family $\pi\colon \Xi \to B$ of smooth projective curves of genus g (i.e. for any proper, smooth morphism π whose fibres $C_t = \pi^{-1}(t)$ are smooth projective curves of genus g, for all $t \in B$), the map $B \to \mathcal{M}_g$, $t \mapsto [C_t]$, where $[\cdot]$ denotes the isomorphism class, is a morphism. It is well known that in order to produce a natural compactification $\widehat{\mathcal{M}_g}$ of $\mathcal{M}_g$ one considers families $\pi\colon \Xi \to B$ with B compact, whose singular fibers are stable curves of genus g.

In this subsection we give explicit descriptions as projective algebraic curves of some of the "limits" which obtained in the previous examples.

All of them arise as singular fibers of relatively well-known pencils of generic non-hyperelliptic curves: the *KFT* pencil, the Costa-Izquierdo-Ying pencil and the Wiman-Edge-Petersen pencil. The 1-dimensional families are canonical curves; in the particular case $p = 3$ of the Costa-Izquierdo-Ying pencil, the curves are contained in a quadric in $\mathbb{P}^3(\mathbb{C})$ and for the Wiman-Edge-Petersen pencil, the curves are contained in a del Pezzo surface of degree 5 in $\mathbb{P}^5(\mathbb{C})$. For the sake of completeness we have included detailed descriptions to enjoy the beautiful geometry involved.

Let $\mathbb{P}^n(\mathbb{C})$ be the n-dimensional complex projective space with homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$. Let $x_0, x_1, \ldots, x_n$ be a basis of $(\mathbb{C}^{n+1})^*$ and let $S^d((\mathbb{C}^{n+1})^*)$ the $\mathbb{C}$-vector space of homogeneous forms of degree d. We take ω to be a primitive third root of unity and let G be a finite group. Usually we will denote forms by lower case letters, and use the corresponding capital letter for the hypersurface which is the zero set of the form.

4.2.1. A projective realization for $\mathcal{M}_3(\mathfrak{S}_4, \theta, s)$. Let $\mathfrak{S}_4$ be the symmetric group of degree 4. There are two irreducible representations of degree 3: one is the standard representation of $\mathfrak{S}_4$ in $\mathbb{C}^3$ and the other is its tensor product with the non-trivial representation of degree 1 of $\mathfrak{S}_4$. These two representations are projectively equivalent. The homomorphism $\rho\colon \mathfrak{S}_4 \to PGL(3, \mathbb{C})$ determined by $\rho(1234) = A$ and $\rho(12) = B$, where

$$A(x_0, x_1, x_2) = (-x_1, x_0, -x_2), \quad B(x_0, x_1, x_2) = (x_0, x_2, x_1),$$

gives a faithful representation of $\mathfrak{S}_4$.

The plane quartics F and N determined by the forms

$$f(x_0, x_1, x_2) = x_0^4 + x_1^4 + x_2^4 \quad \text{and} \quad n(x_0, x_1, x_2) = x_0^2x_1^2 + x_1^2x_2^2 + x_2^2x_0^2$$

are $\rho(\mathfrak{S}_4)$-invariant. Therefore for any generic curve $C_t = F + tN$, $t \in \mathbb{P}^1(\mathbb{C})$, we have that $\rho(\mathfrak{S}_4) \subseteq \mathrm{Aut}(C_t)$.

The pencil $\{\, F + tN : t \in \mathbb{P}^1(\mathbb{C}) \,\}$ of plane quartics is known as the *KFT* pencil and each generic curve C_t of the pencil determines a non-hyperelliptic Riemann surface S_t which is contained in $\mathcal{M}_3(\mathfrak{S}_4, \theta, s)$.

The singular fibers of *KFT* are well known [18] and each Riemann surface with nodes $L \in \mathcal{M}_3(\mathfrak{S}_4, \theta, s)$ will be realized as a singular fiber of *KFT*.

(1) Let $L \in \mathcal{M}_3(\mathfrak{S}_4, \theta, s) \cap (\widehat{\mathcal{M}_3} - \mathcal{M}_3)$ corresponding to $\beta \in \mathrm{Aut}^+(\mathfrak{D})$ be as §4.1.1 case (3). Then L is realized as the curve

$$C_\infty = \{\, (x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{C}) : x_0^2x_1^2 + x_1^2x_2^2 + x_2^2x_0^2 = 0 \,\},$$

which is an irreducible quartic with three double points:

$$e_1 = (1 : 0 : 0), \quad e_2 = (0 : 1 : 0), \quad e_3 = (0 : 0 : 1).$$

The graph of L has a vertex and three loops.

(2) Let $L \in \mathscr{M}_3(\mathfrak{S}_4, \theta, s) \cap (\widehat{\mathscr{M}_3} - \mathscr{M}_3)$ corresponding to $\delta \in \mathrm{Aut}^+(\mathfrak{D})$ as defined in §4.1.1 case (2). Then L is realized as the curve C_{-1} which is the union of two conics

$$C^1 : x_0^2 + \omega x_1^2 + \omega^2 x_2^2 = 0, \qquad C^2 : x_0^2 + \omega^2 x_1^2 + \omega x_2^2 = 0.$$

The curve C_{-1} has 4 double points:

$$p_1 = (1, -1, 1), \quad p_2 = (1, 1, -1),$$
$$p_3 = (-1, 1, 1), \quad p_4 = (1, 1, 1).$$

The graph is the 4-dipole.

(3) Let $L \in \mathscr{M}_3(\mathfrak{S}_4, \theta, s) \cap (\widehat{\mathscr{M}_3} - \mathscr{M}_3)$ corresponding to $\mathrm{Id} \in \mathrm{Aut}^+(\mathfrak{D})$ as defined in §4.1.1 case (1).

We consider the following linear forms:

$$\ell_1(x_0, x_1, x_2) = x_0 + x_1 - x_2, \quad \ell_2(x_0, x_1, x_2) = x_0 - x_1 + x_2,$$
$$\ell_3(x_0, x_1, x_2) = x_0 + x_1 + x_2, \quad \ell_4(x_0, x_1, x_2) = -x_0 + x_1 + x_2.$$

Then L is realized as the curve C_{-2} which is the union of the 4 projective lines determined by ℓ_1, ℓ_2, ℓ_3 and ℓ_4. Each line intersects the three other lines, therefore C_{-2} is a reducible quartic with 6 double points.

The graph $\mathscr{G}(C_{-2})$ can be identified with the 1-skeleton of a tetrahedron.

It can be remarked that the curve $C_2 = \{ (x_0 : x_1 : x_2) : (x_0^2 + x_1^2 + x_2^2)^2 = 0 \}$ corresponds to the unique hyperelliptic curve of genus $g = 3$ with $\mathfrak{S}_4$ as reduced group of automorphisms, which is the point where the 1-dimensional family cuts the sublocus of hyperelliptic curves of genus $g = 3$.

4.2.2. A projective realization for $\mathscr{M}_4(\mathfrak{D}_3 \times \mathfrak{D}_3, \theta, s)$. We consider two copies of the dihedral group $\mathfrak{D}_3$:

$$\mathfrak{D}_3 = \langle a, u : a^3 = 1,\ u^2 = 1,\ uau = a^{-1} \rangle,$$
$$\mathfrak{D}_3 = \langle b, v : b^3 = 1,\ v^2 = 1,\ vbv = b^{-1} \rangle.$$

The homomorphism $\Phi : \mathfrak{D}_3 \times \mathfrak{D}_3 \to GL(4, \mathbb{C})$ determined by

$$\Phi((a, 1)) = A, \quad \Phi((s, 1)) = U, \quad \Phi((1, b) = B, \quad \Phi((1, t)) = V,$$

where

$$A(x_0, x_1, x_2, x_3) = (x_0, \omega x_1, \omega^2 x_2, x_3),$$
$$B(x_0, x_1, x_2, x_3) = (\omega x_0, x_1, x_2, \omega^2 x_3),$$
$$U(x_0, x_1, x_2, x_3) = (-x_0, x_2, x_1, -x_3),$$
$$V(x_0, x_1, x_2, x_3) = (x_3, -x_1, -x_2, x_0),$$

with ω a 3-root of the unity, gives a faithful representation of $\mathfrak{D}_3 \times \mathfrak{D}_3$.

The previous representation Φ induces a representation on $S^d((\mathbb{C}^4)^*)$. We determine the invariant subspaces of $S^2((\mathbb{C}^4)^*)$ and $S^3((\mathbb{C}^4)^*)$ for this representation. A calculation gives the following.

LEMMA 4.2. *The $\Phi(\mathfrak{D}_3 \times \mathfrak{D}_3)$-invariant quadratic and cubic forms are:*

$$q_1 = x_0x_3, \quad q_2 = x_1x_2,$$
$$f_1 = x_0^3 - x_3^3, \quad f_2 = x_1^3 - x_2^3, \quad f_3 = x_3^3 - x_0^3, \quad f_4 = x_2^3 - x_1^3.$$

The image of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ under the Segre embedding

$$\Psi_{(1,1)}\colon (s_0 : s_1) \times (t_0 : t_1) \to (x_0 = s_0t_0 : x_1 = s_0t_1 : x_2 = s_1t_0 : x_3 = s_1t_1)$$

is a quadric smooth hypersurface $Q \subset \mathbb{P}^3(\mathbb{C})$ given as the zero set of the quadratic $(\mathfrak{D}_3 \times \mathfrak{D}_3)$-invariant form $q = q_1 - q_2$.

Any curve C in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \cong Q$ is given by a bihomogeneous polynomial $P(s_0, s_1; t_0, t_1)$. The polynomials

$$P_1(s_0, s_1; t_0, t_1) = s_0^3t_0^3 - s_1^3t_1^3, \quad P_2(s_0, s_1; t_0, t_1) = s_0^3t_1^3 - s_1^3t_0^3$$

determine via the Segre embedding $\Psi_{(1,1)}$ two cubic forms which correspond to f_1 and f_2. Therefore we have a pencil

$$\mathscr{C}_{(a,b)} = \left\{ a(s_0^3t_0^3 - s_1^3t_1^3) + b(s_0^3t_1^3 - s_1^3t_0^3) : (a, b) \in \mathbb{P}^1(\mathbb{C}) \right\}$$

of curves $C_{(a,b)}$ of genus 4 and degree 6 contained in $Q \subset \mathbb{P}^3(\mathbb{C})$ which are $\Phi(\mathfrak{D}_3 \times \mathfrak{D}_3)$-invariant.

Interchanging the two factors of the Segre embedding, we obtain a change of coordinates in $\mathbb{P}^3(\mathbb{C})$. The quadric Q is invariant under this change of coordinates and this exchange of factors corresponds to the transformation $(a, b) \to (a, -b)$. Therefore the map to the moduli space factors over the quotient by this involution.

We consider the singular fibers of the pencil:

$$(a, b) = (1 : 1), \quad (a, b) = (1 : -1), \quad (a, b) = (1 : 0), \quad (a, b) = (0 : 1).$$

First consider one of the cases $(1, 1)$ and $(1, -1)$, which are equivalent. The polynomial $P_1 - P_2$ factors as $(s_0^3 + s_1^3)(t_0^3 - t_1^3)$ and the curve $C_{(1,-1)}$ is the union of six lines, three in each ruling. There are 9 intersections points and therefore the graph $\mathscr{G}(C_{(1,-1)})$ is isomorphic to the complete bipartite graph $K_{3,3}$.

For $(a, b) = (1 : 0)$, the polynomial $P_1 = (s_0^3 t_0^3 - s_1^3 t_1^3)$ factorizes as

$$P_1 = (s_0^3 t_0^3 - s_1^3 t_1^3) = (s_0 t_0 - s_1 t_1)(s_0 t_0 - \omega s_1 t_1)(s_0 t_0 - \omega^2 s_1 t_1).$$

The curve $C_{(1,0)}$ is the union of the following 3 conics contained in Q

$$B_1 = \{s_0 t_0 - s_1 t_1 = 0\},\ B_2 = \{s_0 t_0 - \omega s_1 t_1 = 0\},\ B_3 = \{s_0 t_0 - \omega^2 s_1 t_1 = 0\}$$

intersecting at the points $p_1 = (1 : 0 : 0 : 1)$ and $p_2 = (0 : 1 : 1 : 0)$.

The curve $C_{(1,0)}$ has two D_4 singularities (two triple points) and is not a Riemann surface with nodes. Nevertheless an appropriate stable reduction of the family enables one to obtain a central fiber which is the nodal union of two elliptic components along three points. Therefore its associated graph is, as in §4.1.2 case (2), the 3-dipole.

The projective realization of the Costa-Izquierdo-Ying pencil for any prime $p > 3$ and a possible projective realization of 1-complex dimensional equisymmetric families of type $\mathcal{M}_4(\mathfrak{D}_n \times \mathfrak{D}_n, \theta, s)$, for n odd, will be done elsewhere.

4.2.3. A projective realization for $\mathcal{M}_6(\mathfrak{A}_5, \theta, s)$. A plane sextic which has 4 double points at 4 points in general position determines a non-hyperelliptic curve of genus $g = 6$. It is well known that the blowing up of $\mathbb{P}^2(\mathbb{C})$ centered at 4 points in general position is a unique (modulo $\mathrm{Aut}(\mathbb{P}^2(\mathbb{C}))$) del Pezzo surface F of degree 5 in $\mathbb{P}^5(\mathbb{C})$. The group of automorphisms of F is isomorphic to $\mathfrak{S}_5$. In [12], Edge describes the WEP pencil explicitly as a family of quadric sections of the del Pezzo surface F where a linear representation of $\mathfrak{A}_5$ acts. We will follow his description.

We will denote by $(y_0 : y_1 : \cdots : y_5)$ the homogeneous coordinates of the complex projective space $\mathbb{P}^5(\mathbb{C})$. Let Q and Ω be the following quadrics:

$$Q = \sum_{i=0}^{5} y_i^2 + (y_0+y_1+y_2)^2 + (y_0+y_4-y_5)^2 + (y_1+y_5-y_3)^2 + (y_2+y_3-y_4)^2,$$
$$\Omega = y_3 y_0 + y_4 y_1 + y_5 y_2.$$

Let $\mathscr{P}_F$ be the pencil of canonical curves of genus $g = 6$ on the del Pezzo quintic surface F. This pencil is the image of a pencil of plane sextics under the map:

$$\begin{aligned} y_0 &= (x_1 + x_2)^2 (x_1 - x_2), & y_1 &= (x_2 + x_0)^2 (x_2 - x_0), \\ y_2 &= (x_0 + x_1)^2 (x_0 - x_1), & y_3 &= (x_1 + x_2)(x_1 - x_2)^2, \\ y_4 &= (x_2 + x_0)(x_2 - x_0)^2, & y_5 &= (x_0 + x_1)(x_0 - x_1)^2. \end{aligned}$$

All these sextics pass through all the points of the subset

$$\Delta = \{ q_1 = (1:1:1),\ q_2 = (1:1:-1),\\ q_3 = (1:-1:1),\ q_4 = (-1:1:1) \}.$$

The cutting of each curve on F with the quadric $Q + t\Omega$, $t \in \mathbb{P}^1(\mathbb{C})$ is invariant under $\mathfrak{A}_5$. Cuttings by the two quadrics $Q \pm t\Omega$ are transposed by the operations of $\mathfrak{S}_5/\mathfrak{A}_5$, see [12, p. 245].

The two exceptions, invariant under the whole group $\mathfrak{S}_5$, are $t = 0$, which corresponds to the Wiman sextic

$$\Gamma\colon 2x_0^6 + 2x_1^6 + 2x_2^6 + 2(x_0^2 + x_1^2 + x_2^2)(x_0^4 + x_1^4 + x_2^4) - 24x_0^2x_1^2x_2^2 = 0,$$

and $t = \infty$, which corresponds to the sextic

$$\Pi\colon 3(x_1^2 - x_2^2)(x_2^2 - x_0^2)(x_0^2 - x_1^2) = 0.$$

The curves of the pencil $\mathscr{P}_F$ other than Π, are all canonical curves of genus $g = 6$, except for a pair Σ^+, Σ^- each composed of five conics and a pair R^+, R^- of rational curves each with six nodes. The values of t for the pair Σ^+, Σ^- are $t = \pm 2i/\sqrt{3}$ and for the pair R^+, R^- they are $t = \pm(10\sqrt{5})/3$.

In the complex projective plane we consider the four lines

$$M_1\colon x_0 + \omega x_1 + \omega^2 x_2 = 0, \quad M_2\colon x_0 + \omega x_1 - \omega^2 x_2 = 0,\\ M_3\colon x_0 - \omega x_1 + \omega^2 x_2 = 0, \quad M_4\colon x_0 - \omega x_1 - \omega^2 x_2 = 0$$

and the conic $C\colon x_0^2 + \omega^2 x_1^2 + \omega x_2^2 = 0$.

The union of the 4 lines M_i and the conic C is a plane sextic M which passes through the 4 points q_i of Δ and corresponds to Σ^+. The sextic M has also 4 nodes r_i, $1 \le i \le 4$, which are the intersection points of each line M_i, $1 \le i \le 4$, with C that are not contained in Δ and 6 nodes m_{ij}, $i > j$, which are the intersection points of the lines.

The sextic N corresponding to the rational curve R^+ has 6 nodes:

$$n_1 = (0:\tau:1), \quad n_2 = (\tau:1:0), \quad n_3 = (1:0:\tau),\\ n_4 = (0:-\tau:1), \quad n_5 = (-\tau:1:0), \quad n_6 = (1:0:-\tau),$$

where $\tau = (1+\sqrt{5})/2$.

(1) Let $L \in \widehat{\mathscr{M}_6}(\mathfrak{A}_5, \theta; s) \cap (\widehat{\mathscr{M}_6} - \mathscr{M}_6)$ corresponding to $\delta = \mathrm{Id} \in \mathrm{Aut}^+(\mathfrak{D})$ as in §4.1.3 case (1). In that case L is realized as the strict transform Σ^+ of M under the blowing up of $\mathbb{P}^2(\mathbb{C})$ centered at the 4 points q_i, $1 \le i \le 4$. The graph $\mathscr{G}(\Sigma^+)$ is isomorphic to the complete graph K_5.

(2) Let $L \in \widehat{\mathscr{M}_6}(\mathfrak{A}_5, \theta; s) \cap (\widehat{\mathscr{M}_6} - \mathscr{M}_6)$ corresponding to $\delta \in \mathrm{Aut}^+(\mathfrak{D})$ in §4.1.3 case (2). In that case, L is realized as the strict transform R^+ under the blowing up of $\mathbb{P}^2(\mathbb{C})$ centered at the 4 points q_i, $1 \leq i \leq 4$, of the irreducible sextic N. The graph $\mathscr{G}(R^+)$ has 1 vertex and 6 loops corresponding to the 6 double points.

(3) Let $L \in \widehat{\mathscr{M}_6}(\mathfrak{A}_5, \vartheta; s) \cap (\widehat{\mathscr{M}_6} - \mathscr{M}_6)$ as in §4.1.3 case (3). In that case, L is realized as the strict transform Ω of Π under the blowing of $\mathbb{P}^2(\mathbb{C})$ centered at the 4 points q_i, $1 \leq i \leq 4$. The graph $\mathscr{G}(\Omega)$ is isomorphic to the Petersen graph and admits $\mathfrak{S}_5$ as automorphism group.

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