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AUTOMORPHISMS OF THE MODULI SPACE OF PRINCIPAL G-BUNDLES INDUCED BY OUTER AUTOMORPHISMS OF G

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Abstract

In this work we study finite-order automorphisms of the moduli space of principal G-bundles coming from outer automorphisms of the structure group when G is a simple complex Lie group. We do this by describing the subvarieties of fixed points for the action of that automorphisms on the moduli space of principal G-bundles. In particular, we prove that these fixed points are reductions of structure group to the subgroup of fixed points of the outer automorphism. Moreover, we study the way in which these fixed points fall into the stable or nonstable locus of the moduli.

1. Introduction

Let X be a smooth complex projective irreducible curve of genus $g \geq 2$ and let G be a complex reductive Lie group. Principal G-bundles over X are geometric objects which come with an action of the group of transformations or structure group, G. Geometric properties of principal G-bundles depend on the structure group G. Using Mumford's GIT [13], Ramanathan [17], [18] and [19] gave a notion of stability for principal G-bundles in order to construct the moduli space of polystable principal G-bundles, $M(G)$, as a coarse moduli scheme whose points correspond bijectively to isomorphism classes of polystable G-bundles and whose open subset of non-singular points corresponds exactly to the set of stable bundles. With this construction, the classical theorem of Narasimhan and Seshadri [14] remains true for a semisimple Lie group G: if K is a maximal compact subgroup of G, there is a bijection between conjugacy classes of representations $\pi_1(X) \to K$ and isomorphism classes of G-bundles.

These spaces have a very rich topology and geometry and have been intensively studied in mathematics and theoretical physics. A way to study the moduli space $M(G)$ is to describe the subvarieties and automorphisms of $M(G)$, in the spirit of Serman [21] and Kouvidakis and Pantev [12]. Serman [21], for example, proves that the forgetful morphisms $M(\mathrm{O}(n, \mathbb{C})) \to M(\mathrm{GL}(n, \mathbb{C}))$ and $M(\mathrm{Sp}(2n, \mathbb{C})) \to M(\mathrm{SL}(2n, \mathbb{C}))$ are closed immersions, so they define

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subvarieties which are isomorphic to $M(\mathrm{O}(n, \mathbb{C}))$ and $M(\mathrm{Sp}(2n, \mathbb{C}))$ respectively.

In our work we consider subvarieties of $M(G)$ defined by fixed points of certain finite-order automorphisms of $M(G)$. This connects with the work of Kouvidakis and Pantev [12]. They determined the group of automorphisms of the moduli space in the particular case of vector bundles with a given rank and degree. They proved that any automorphism of $M(\mathrm{SL}(n, \mathbb{C}))$ is a combination of the automorphisms of the moduli space which comes from the outer involution of the Dynkin diagram, $E \mapsto E^*$, with the automorphisms which consists on tensoring by a line bundle of order n and the automorphisms which come from automorphisms of the base curve, X.

In [11] the same result is proved with other techniques that involve Hecke curves, which are minimal rational curves coming from Hecke transformations. Other authors have recently worked along this path dealing with other particular groups (for example, Biswas, Gómez and Muñoz [4] and [5]). Using the perspective of [11], Biswas, Gómez and Muñoz [5] give a complete description of the group of automorphisms of $M(\mathrm{Sp}(2n, \mathbb{C}))$. They prove that these automorphisms combine automorphisms of the base curve with those induced by line bundles of order two. Observe that in this case there are no automorphisms of the moduli space coming from outer automorphisms of the group because the Dynkin diagram has no symmetries.

In general, given a complex semisimple Lie group G, the group of automorphisms of $M(G)$ contains elements coming from the action of the center of G on the moduli space, the action of $\mathrm{Out}(G)$ and those induced by $\mathrm{Aut}(X)$ (these are the only automorphisms in the case of vector and symplectic bundles; the general case is open, as far as we know). Here, we will pay attention to a family of simple complex Lie groups G and those automorphisms of $M(G)$ coming from the action of $\mathrm{Out}(G)$ on $M(G)$. Of course, this family of automorphisms does not constitute the whole group of automorphisms of $M(G)$, but it will help us to deepen the study of the geometry of $M(G)$ by describing certain subvarieties of fixed points of the moduli and relating them to its singular locus. In [2], a complete description of the action of $\mathrm{Out}(G)$ on $M(G)$ is provided, which we will recall here. Here we further that work, relate it to the strictly polystable locus of $M(G)$ and deal with other interesting groups which have not been treated in the cited works. We also describe the automorphisms of $M(G)$ coming from outer automorphisms of the structure group G in the case when G is simple.

We will see how the group of outer automorphisms $\mathrm{Out}(G)$ acts in $M(G)$ for any semisimple complex Lie group G and describe the subvariety of fixed points of these finite order automorphisms when the group $\mathrm{Out}(G)$ is not trivial, that is, when G is of type A_n, D_n or E_6. We give a full description of the subvarieties of fixed points of these outer automorphisms, including a discussion of whether the corresponding bundles are stable or not. We prove that the announced fixed points are reductions of the structure group to the subgroup of fixed points of the outer automorphism of G. We will also see that only A_n and D_n for $n \geq 4$ admit stable fixed points for their outer involutions and that in the case of D_4, fixed points of triality automorphism are also fixed points for the remaining outer involutions of the moduli. In the case of D_n, we describe the nonstable locus as the union of n irreducible components, the nonstable fixed points of the outer involution of the moduli being the union of $n-1$ of those components. When $n=4$, fixed points for the action of triality are always strictly polystable and fall exactly into one of those irreducible components, so they are also fixed for the remaining outer involutions of the moduli. Finally, we describe precisely fixed points of the only outer involution of $M(E_6)$ as reductions of the structure group to F_4. We also see that these fixed points are always strictly polystable. In all the cases, we study what happens with the isogenous groups and explain it in terms of the situation for the corresponding simply connected group.

The paper is organized as follows. In Section 2, we recall the notion of principal bundle, stability and its formulation in terms of filtrations in order to give a reduced useful notion of stability for E_6-bundles. In Section 3, we define and describe the action of $\mathrm{Out}(G)$ on $M(G)$ for a semisimple complex Lie group G. Sections 4 and 5 are devoted to develop the cases of A_n and D_n, respectively. The algebraic foundations of exceptional groups, particularly of F_4 and E_6, are explained in Section 6. Finally, in Section 7 we prove the main results about fixed points of the outer involution of $M(E_6)$. This case is the least discussed in the literature, as far as we know.

2. The moduli space of principal G-bundles

Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. Let X be a compact complex algebraic curve of genus g. It is well-known that a principal G-bundle over X is a complex manifold, E, equipped with a holomorphic projection map $\pi\colon E \to X$ and a holomorphic right action of G on E which preserves the projection π. If we denote by $\underline{G}$ the natural sheaf on X induced by G, the set of isomorphism classes of principal G-bundles is parameterized by $H^1(X, \underline{G})$. The well-known case of vector bundles is obtained from this by taking $G = \mathrm{GL}(n, \mathbb{C})$.

A general notion of stability for principal G-bundles (and in fact for general G-pairs), involving parabolic subgroups and characters, is studied in [9] following [18]. We briefly recall it here and explain it in terms of filtrations.

Let E be a principal G-bundle. Choose H a maximal compact subgroup of G with Lie algebra $\mathfrak{h}$. We fix a faithful holomorphic representation $\rho\colon G \to$

$\mathrm{GL}(W)$ of G coming from a unitary representation of H. Call $E(W)$ the induced vector bundle. Take P a parabolic subgroup of G and σ a reduction of structure group of E to P. An antidominant character, χ, of P gives rise, via the Killing form, to an element $s_\chi \in i\mathfrak{h}$. The endomorphism $\rho(s_\chi)$ diagonalizes with real eigenvalues $\lambda_1 < \cdots < \lambda_k$, giving rise to a decomposition $E(W) = \bigoplus_{i=1}^k F_i$, where the restriction of $\rho(s_\chi)$ to F_i is just the multiplication by λ_i. This induces a filtration of E by holomorphic vector subbundles

$$\mathscr{F}_{\sigma,\chi} \equiv 0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = E(W), \tag{1}$$

where $E_i = \bigoplus_{j=1}^i F_j$. For this filtration $\mathscr{F}_{\sigma,\chi}$ we have an associated degree,

$$\deg \mathscr{F}_{\sigma,\chi} = \lambda_k \deg E(W) + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg E_i.$$

Definition 2.1. Let E be a principal G-bundle, $\rho \colon G \to \mathrm{GL}(W)$ a faithful holomorphic representation of G, P a parabolic subgroup of G and σ a reduction of structure group of E to P. Let χ be an antidominant character of P. Let $\mathscr{F}_{\sigma,\chi}$ be the filtration defined in (1). Then, the bundle E is *stable* (resp. *semistable*) if

$$\deg \mathscr{F}_{\sigma,\chi} = \lambda_k \deg E(W) + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg E_i > 0 \quad (\text{resp.} \geq 0)$$

for every filtration and sequence of weights coming from elements P, σ, χ as above. It is *polystable* if it is semistable and for each P, σ, χ for which $\deg \mathscr{F}_{\sigma,\chi} = 0$, E admits a reduction of the structure group to a Levi subgroup, L, of P.

It is also known that it suffices to check this for maximal parabolic subgroups.

For example, if $G = \mathrm{SL}(n, \mathbb{C})$, E is naturally a rank n complex vector bundle with trivial determinant bundle. Filtrations of the type considered above for maximal parabolic subgroups correspond exactly with choices of a vector subbundle F of E. More explicitly, we consider the fundamental representation

$$\rho \colon G \to \mathrm{GL}(n, \mathbb{C}),$$

where $W = \mathbb{C}^n$. If E is a principal G-bundle, P is a parabolic subgroup of G, σ is a holomorphic reduction of the structure group of E to P and χ is an antidominant character of P, then P, σ, χ induce a filtration of holomorphic vector bundles of $E(W)$,

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = E(W),$$

and an increasing sequence of weights,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k,$$

such that

$$\deg E(\sigma, \chi) = \lambda_k \deg E(W) + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg E_{\leq i}.$$

Since $G = \mathrm{SL}(n, \mathbb{C})$, we cover all possible filtrations of E and sequences of weights of that form (see [15, Chapter 6]).

If $G = \mathrm{SO}(n, \mathbb{C})$, the parabolic subgroups correspond to filtrations of the form

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = E(W)$$

with $V_i^{\perp} = V_{k-i}$ for all $i = 0, \ldots, k$ (observe that, in this case, $E(W)$ is a vector bundle equipped with a non degenerate bilinear form) and weights of the form

$$-\lambda_r < \cdots < -\lambda_1 \leq \lambda_1 < \cdots < \lambda_r.$$

Therefore, for $G = \mathrm{SO}(n, \mathbb{C})$, $n \geq 3$, maximal parabolic subgroups correspond to choices of an isotropic subbundle of E and sequences of weights of the form

$$-\lambda < \lambda$$

with $\lambda > 0$.

Some words for the case of the group $\mathrm{Spin}(n, \mathbb{C})$, the complex simply-connected Lie group with Lie algebra $\mathfrak{so}(n, \mathbb{C})$. It is a double cover of the group $\mathrm{SO}(n, \mathbb{C})$. This means that every principal Spin-bundle gives rise to a principal SO-bundle via the projection $\pi \colon \mathrm{Spin}(n, \mathbb{C}) \to \mathrm{SO}(n, \mathbb{C})$ and a principal Spin-bundle is stable (resp. semistable, polystable) if and only if the corresponding SO-bundle is so. This is because the projection π gives a bijective correspondence between parabolic and Levi subgroups of $\mathrm{Spin}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$ which leaves invariant the degrees considered in the study of stability in terms of filtrations.

The moduli space of principal G-bundles is then an algebraic variety which parametrizes isomorphism classes of polystable principal G-bundles. From results of Ramanathan in [18] and [19], we know that the dimension of $M(G)$ is $\dim \mathfrak{g}(g-1)$, where $\mathfrak{g}$ is the Lie algebra of G and that there is a bijective correspondence between $\pi_0(M(G))$ and $\pi_1(G)$. We will deal with the case in which G is simply connected, so in this case $M(G)$ is irreducible. For more details and a further study see [10], where the construction of moduli of bundles over more general varieties is treated.

An important notion that plays a role in the study of the topology of the moduli space of principal bundles and is relevant in our analysis of fixed points is the notion of simplicity.

DEFINITION 2.2. A principal G-bundle E is called *simple* if the only automorphisms of E are those coming from the centre of the structure group.

It is well-known that stability implies simplicity for vector bundles. This is not true for an arbitrary semisimple or simple complex Lie group G. In [16, Proposition 4.5], for example, Ramanan constructs stable orthogonal bundles which are not simple.

Stability and simplicity establish open conditions on $M(G)$. In fact, we have the following well-known result.

PROPOSITION 2.3. *Let G be a complex reductive Lie group G and let $M(G)$ be the moduli space of principal G-bundles. Then, the open subvariety of stable and simple bundles in $M(G)$ is contained in the smooth locus of the moduli space.*

This is what we can say for a general semisimple group. In the case of vector bundles, stability implies simplicity, so this open subvariety coincides with the open subset of stable bundles. In fact, in this case strictly polystable vector bundles are always bundles representing singular points of the moduli (except when $g = 2$ and $n = 2$), so the smooth locus is exactly the open subset of stable bundles.

3. The action of $\mathbf{Out}(G)$ on $M(G)$

Let G be a semisimple complex Lie group. In order to study automorphisms of $M(G)$, we consider the action of the group $\mathrm{Aut}(G)$ of automorphisms of the Lie group G on the set of isomorphism classes of principal G-bundles over X in the following way (see [2]).

DEFINITION 3.1. Let E be a principal G-bundle. If $A \in \mathrm{Aut}(G)$ and $\{(U_i, \varphi_i)\}_i$ is a trivializing covering of E, then $\{(U_i, \mathrm{id}_{U_i} \times A \circ \varphi_i)\}_i$ is a trivializing cover of a certain principal G-bundle, where

$$\pi^{-1}(U_i) \xrightarrow{\varphi_i} U_i \times G \xrightarrow{\mathrm{id}_{U_i} \times A} U_i \times G.$$

We define $A(E)$ to be this principal G-bundle.

In fact, if $\{\psi_{ij}\}_{ij}$ is a family of transition functions of E associated to the covering given in the definition, the transition functions of the new bundle $A(E)$ associated to this covering are $\{A \circ \psi_{ij}\}_{ij}$.

It is clear that E and $A(E)$ are isomorphic when A is an inner automorphism, so inner automorphisms induce the identity map on the set of isomorphism classes of principal G-bundles.

Then, we have that the group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$ acts on the set of isomorphism classes of principal G-bundles over X in the following way: if $\sigma \in \mathrm{Out}(G)$ and $A \in \mathrm{Aut}(G)$ is an automorphism of G representing σ, then $\sigma(E) = A(E)$. Moreover, an automorphism of G gives rise to a bijection between parabolic subgroups and characters of these parabolics of G and between filtrations as explained above, preserving the degrees. Then, $\sigma(E)$ is stable (resp. semistable, polystable) if E is so. This proves that this action in fact induces an action of $\mathrm{Out}(G)$ on the moduli space $M(G)$.

The following equivalence relation on $\mathrm{Aut}(G)$ will also be relevant in our analysis of fixed points.

Definition 3.2. If $\alpha, \beta \in \mathrm{Aut}(G)$, we say that $\alpha \sim_i \beta$ if there exists $\theta \in \mathrm{Int}(G)$ such that $\alpha = \theta \circ \beta \circ \theta^{-1}$.

We have the following result.

Lemma 3.3. *Let* $\alpha, \beta \in \mathrm{Aut}(G)$. *If* $\alpha \sim_i \beta$*, then* α *and* β *define the same element in* $\mathrm{Out}(G)$.

Proof. If $\alpha \sim_i \beta$, then there exists $\sigma \in \mathrm{Int}(G)$ such that $\alpha = \sigma\beta\sigma^{-1}$. Then $\alpha\beta^{-1} = \sigma\beta\sigma^{-1}\beta^{-1}$. Since $\mathrm{Int}(G)$ is a normal subgroup, $\tau = \beta\sigma^{-1}\beta^{-1} \in \mathrm{Int}(G)$, so $\alpha\beta^{-1} = \sigma\tau$. From this, $\alpha = (\sigma\tau)\beta$ or, equivalently, $\alpha \sim \beta$.

Given an outer automorphism α of G, two automorphisms a_1 and a_2 of G representing the element α of $\mathrm{Out}(G)$ have the same subgroup of fixed points if $a_1 \sim_i a_2$. This is the importance of the equivalence relation $\sim_i$. In [23] and [24], Wolf and Gray studied these possible fixed points subgroups for a reductive Lie group G, so we will make use of these useful results in our study of fixed points of the action of outer automorphisms in $M(G)$ when G is simple.

Let $\mathfrak{g}$ be a simple complex Lie algebra and let G be the simply-connected complex Lie group with Lie algebra $\mathfrak{g}$. In this case, we know that $\mathrm{Out}(G) \cong \mathrm{Out}(\mathfrak{g})$. It is also well known that, since $\mathfrak{g}$ is simple, there is a natural isomorphism between symmetries of the Dynkin diagram of $\mathfrak{g}$ and its group of outer automorphisms. Finally, we have in this case a complete description of $\mathrm{Out}(G)$, given by the group of symmetries of the Dynkin diagram. For this reason, from now on we will work primarily with this kind of group.

Finally, observe that, in the simple case, $\mathrm{Out}(G)$ is not trivial only when G is of type A_n, D_n or E_6. Indeed, we will deal with these three cases.

4. The case of A_n

Take $\mathfrak{g}$ a classical simple complex Lie algebra of type A_n with $n \geq 2$ and G the simply-connected complex Lie group with Lie algebra $\mathfrak{g}$, $G = \mathrm{SL}(n+1, \mathbb{C})$. The Dynkin diagram of A_n is of the form

The group of outer automorphisms of $\mathfrak{g}$ is isomorphic to the group of symmetries of the Dynkin diagram. Then, $\mathrm{Out}(G) \cong \mathbb{Z}_2$, so the action of the group of outer automorphisms induces an involution σ of the moduli space $M(G)$ of polystable rank $n+1$ complex vector bundles with trivial determinant. This involution acts by sending the bundle E to its dual, E^*,

$$\sigma(E) = E^*. \tag{2}$$

The following result gives a complete description of fixed points in $M(G)$ for the action of the involution σ.

PROPOSITION 4.1. *Let σ be the involution of $M(\mathrm{SL}(n+1, \mathbb{C}))$ induced by the outer involution of A_n. Let E be a polystable principal* $\mathrm{SL}(n+1, \mathbb{C})$*-bundle. Then, E is fixed for the action of σ if and only if E admits a reduction of structure group to one of the following subgroups of* $\mathrm{SL}(n+1, \mathbb{C})$*:*

(1) $\mathrm{SO}(n+1, \mathbb{C})$*,*

(2) $\mathrm{Sp}(n+1, \mathbb{C})$*, which is only possible if $n+1$ is even,*

(3) $\mathrm{SO}(p, \mathbb{C}) \times \mathrm{Sp}(n-p+1, \mathbb{C})$*, for some p with $1 \leq p \leq n$ and $n-p+1$ even.*

PROOF. Suppose that E is a fixed point of σ. Then, by (2) there exists an isomorphism $f : E \to E^*$. It is then clear that $(f^t)^{-1} \circ f$ is an automorphism of E. If we take E to be stable, then the following must hold

$$(f^t)^{-1} \circ f = \lambda I \tag{3}$$

for some $\lambda \in \mathbb{C}^*$ with $\lambda^{n+1} = 1$. From (3), we have that $f = \lambda f^t$. But we have also that $f = \lambda^{-1} f^t$, by transposing (3), so we conclude that $\lambda = \pm 1$. The case $\lambda = -1$ is only possible if the rank of E is even, say $n+1 = 2m$.

If $\lambda = 1$, then f defines a non-degenerate symmetric bilinear form, so it induces a reduction of structure group of E to $\mathrm{SO}(n+1, \mathbb{C})$. In the even rank case, if $\lambda = -1$, f defines a symplectic form, so E reduces its structure group to $\mathrm{Sp}(n+1, \mathbb{C})$.

If E is polystable, then it is a direct sum of stable vector bundles on which the same reasoning works. So, there are three possibilities:

(1) E admits an orthogonal structure, so E reduces to $\mathrm{SO}(n+1, \mathbb{C})$,

(2) E admits a symplectic structure, so E reduces to $\mathrm{Sp}(n+1, \mathbb{C})$ (this can only happen of course if $n+1 = 2m$),

(3) E is the direct sum of an orthogonal bundle and a symplectic bundle, so it reduces to $\mathrm{SO}(p, \mathbb{C}) \times \mathrm{Sp}(n-p+1, \mathbb{C})$, for some p with $1 \leq p \leq n$ and $n-p+1$ even.

Conversely, every reduction of structure group as above induces an isomorphism $f\colon E \to E^*$, so it is only possible when E is a fixed point of σ.

Recall that for any subgroup H of a reductive complex Lie group G, the inclusion $H \hookrightarrow G$ induces a forgetful map $M(H) \to M(G)$ which is in fact a morphism of the moduli spaces. Take positive integers p, q with $p+q = n+1$ and q even. Then, the maps $M(\mathrm{SO}(p, \mathbb{C})) \to M(\mathrm{SL}(n+1, \mathbb{C}))$ and $M(\mathrm{Sp}(q, \mathbb{C})) \to M(\mathrm{SL}(n+1, \mathbb{C}))$ give rise to a morphism

$$M(\mathrm{SO}(p, \mathbb{C})) \times M(\mathrm{Sp}(q, \mathbb{C})) \to M(\mathrm{SL}(n+1, \mathbb{C}))$$

defined by taking the direct sum of the bundles.

This allows us to define the following subvarieties of $M(\mathrm{SL}(n+1, \mathbb{C}))$.

Definition 4.2. Let $n \geq 2$ and let $G = \mathrm{SL}(n+1, \mathbb{C})$.

(1) Suppose first that n is odd (so $n+1$ is even). Say $n+1 = 2m$. We define, for $0 \leq r \leq m$,

$$\begin{aligned} N(r) &= \mathrm{Im}\big(M(\mathrm{SO}(2r, \mathbb{C})) \times M(\mathrm{Sp}(2m-2r, \mathbb{C})) \\ &\qquad\qquad\qquad\qquad \to M(\mathrm{SL}(n+1, \mathbb{C}))\big) \\ &= \mathrm{Im}\big(M(D_r) \times M(C_{m-r}) \to M(A_n)\big). \end{aligned}$$

(2) Suppose now that n is even (so $n+1$ is odd), say $n = 2m$. Then, we define for $0 \leq r \leq m$,

$$\begin{aligned} N(r) &= \mathrm{Im}\big(M(\mathrm{SO}(2r+1, \mathbb{C})) \times M(\mathrm{Sp}(2m-2r, \mathbb{C})) \\ &\qquad\qquad\qquad\qquad \to M(\mathrm{SL}(n+1, \mathbb{C}))\big) \\ &= \mathrm{Im}\big(M(B_r) \times M(C_{m-r}) \to M(A_n)\big). \end{aligned}$$

In terms of the definition above, we can rewrite what is proved in Proposition 4.1.

Proposition 4.3. *Let $n \geq 2$ and let $G = \mathrm{SL}(n+1, \mathbb{C})$. Let σ be involution of $M(G)$ coming from the nontrivial outer involution of G. Let $m = \lfloor (n+1)/2 \rfloor$.*

Then, the subvariety of fixed points in $M(G)$ for the action of σ is exactly

$$\bigcup_{r=0}^{m} N(r),$$

where, for $0 \leq r \leq m$, $N(r)$ is determined in Definition 4.2.

We are now in position to study whether the fixed points studied above are stable bundles or not.

PROPOSITION 4.4. *Let $n \geq 2$. Let $G = \mathrm{SL}(n+1, \mathbb{C})$ and σ be the involution of $M(G)$ induced by the outer involution of A_n. Let $\mathrm{Fix}(\sigma)$ be the subvariety of fixed points in $M(G)$ for the action of σ. For $0 \leq r \leq m = \lfloor (n+1)/2 \rfloor$, we define*

$$N_{\mathrm{s}}(m) = \{E \in N(m) : E \textit{ is stable and simple as an } \mathrm{SO}(2m+1, \mathbb{C})\textit{-bundle}\}$$

and

$$N_{\mathrm{ps}}(m) = N(m) \setminus N_{\mathrm{s}}(m).$$

($N(m)$ is specified in Definition 4.2). We also denote by $M^{\mathrm{s}}(G)$ the stable locus of $M(G)$ and by $M^{\mathrm{ps}}(G)$ the strictly polystable locus of $M(G)$. We have the following:

(1) *If $n+1$ is odd, then $\mathrm{Fix}(\sigma) = N_{\mathrm{s}}(m) \cup \big(N_{\mathrm{ps}}(m) \cup \bigcup_{r=0}^{m-1} N(r)\big)$ and*

$$\mathrm{Fix}(\sigma) \cap M^{\mathrm{s}}(G) = N_{\mathrm{s}}(m),$$
$$\mathrm{Fix}(\sigma) \cap M^{\mathrm{ps}}(G) = N_{\mathrm{ps}}(m) \cup \bigcup_{r=0}^{m-1} N(r).$$

(2) *If $n+1$ is even, then $\mathrm{Fix}(\sigma) = \big(N_{\mathrm{s}}(0) \cup N_{\mathrm{s}}(m)\big) \cup \big(N_{\mathrm{ps}}(0) \cup N_{\mathrm{ps}}(m) \cup \bigcup_{r=1}^{m-1} N(r)\big)$ and*

$$\mathrm{Fix}(\sigma) \cap M^{\mathrm{s}}(G) = N_{\mathrm{s}}(0) \cup N_{\mathrm{s}}(m),$$
$$\mathrm{Fix}(\sigma) \cap M^{\mathrm{ps}}(G) = N_{\mathrm{ps}}(0) \cup N_{\mathrm{ps}}(m) \cup \bigcup_{r=1}^{m-1} N(r).$$

PROOF. Suppose first that $n+1$ is odd, that is, $n+1 = 2m+1$ for some m. It is immediate that $N(r)$ falls into the nonstable locus (and, then, in the singular locus) of $M(\mathrm{SL}(n+1, \mathbb{C}))$ when $0 \leq r \leq m-1$. This is not the case when $r = m$. Results by Serman [21] show that the natural algebraic morphism $M(\mathrm{SO}(2m+1, \mathbb{C})) \to M(\mathrm{SL}(2m+1, \mathbb{C}))$ is injective, so we can identify in some sense the image of this map, that is, $N(m)$, with $M(\mathrm{SO}(2m+1,\mathbb{C}))$. Then, in this case we have

$$\begin{aligned} N(m) &= \mathrm{Im}(M(\mathrm{SO}(2m+1,\mathbb{C})) \to M(\mathrm{SL}(2m+1,\mathbb{C}))) \\ &\cong M(\mathrm{SO}(2m+1,\mathbb{C})). \end{aligned}$$

An element of $N(m)$ is stable if it is a special orthogonal bundle stable as a vector bundle. Results by Ramanan [16, Proposition 4.5] show that this is only possible when the bundle is stable and simple and a special orthogonal bundle. Then, the subvariety of fixed points in $M(\mathrm{SL}(2m+1,\mathbb{C}))$ may be decomposed in the following way

$$N_{\mathrm{s}}(m) \cup \left(N_{\mathrm{ps}}(m) \cup \bigcup_{r=0}^{m-1} N(r) \right),$$

where the first part lies in the stable (so smooth) locus of the moduli and the second part lies in the singular locus, as we wanted to prove.

Suppose now that $n+1$ is even, so $n+1=2m$ for some m. As in the odd case, $N(r)$ falls into the nonstable locus of $M(\mathrm{SL}(n+1,\mathbb{C}))$, but this is not the case for $N(0)$ and $N(m)$. Observe that

$$\begin{aligned} N(0) &= M(\mathrm{Sp}(2m,\mathbb{C})), \\ N(m) &= M(\mathrm{SO}(2m,\mathbb{C})). \end{aligned}$$

Similar arguments as given in the odd case work here for SO and Sp in even rank. So we have analogous definitions for $N_{\mathrm{s}}(0)$, $N_{\mathrm{ps}}(0)$, $N_{\mathrm{s}}(m)$ and $N_{\mathrm{ps}}(m)$. We have now that $N_{\mathrm{s}}(0)$ and $N_{\mathrm{s}}(m)$, given by symplectic and orthogonal bundles respectively, lie in the stable (thus smooth) locus of $M(\mathrm{SL}(2m,\mathbb{C}))$ and the rest fall into the nonstable (thus singular) locus. The corresponding decomposition is then the announced:

$$\big(N_{\mathrm{s}}(0) \cup N_{\mathrm{s}}(m)\big) \cup \left(N_{\mathrm{ps}}(0) \cup N_{\mathrm{ps}}(m) \cup \bigcup_{r=1}^{m-1} N(r) \right).$$

Now, let $G = \mathrm{PSL}(n+1,\mathbb{C})$, the centerless complex Lie group with Lie algebra $\mathfrak{sl}(n+1,\mathbb{C})$. From the exact sequence of groups

$$1 \to \mathbb{Z}_{n+1} \to \mathrm{SL}(n+1,\mathbb{C}) \xrightarrow{\pi} \mathrm{PSL}(n+1,\mathbb{C}) \to 1,$$

it is clear that the outer involution σ of $\mathrm{SL}(n+1,\mathbb{C})$ descends to an outer involution

$$\bar{\sigma} \colon \mathrm{PSL}(n+1,\mathbb{C}) \to \mathrm{PSL}(n+1,\mathbb{C}).$$

There is a natural map

$$\rho\colon M(\mathrm{SL}(n+1,\mathbb{C})) \to M(\mathrm{PSL}(n+1,\mathbb{C})).$$

and there is a bijective correspondence between parabolic subgroups of both groups via the projection map π. This says that a principal $\mathrm{SL}(n+1,\mathbb{C})$-bundle is stable (resp. semistable, polystable) if and only if $\rho(E)$ is stable (resp. semistable, polystable).

Recall that, since $Z(\mathrm{SL}(n+1,\mathbb{C})) \cong \mathbb{Z}_{n+1}$, $M(\mathrm{PSL}(n+1,\mathbb{C}))$ has $n+1$ connected components, corresponding to the elements of $Z(\mathrm{SL}(n+1,\mathbb{C}))$, and the image of ρ is exactly one of these connected components. The outer involution $\bar{\sigma}$ permutes the connected components via the action of the outer involution of $\mathrm{SL}(n+1,\mathbb{C})$ on the center.

We can then give an analogous description of fixed points of $\bar{\sigma}$ as in Proposition 4.1.

Proposition 4.5. *Let $\bar{\sigma}$ be the involution of $M(\mathrm{PSL}(n+1,\mathbb{C}))$ induced by the outer involution of A_n. Then,*

(1) *If n is even, $\bar{\sigma}$ has no fixed points,*
(2) *If n is odd, a polystable principal $\mathrm{PSL}(n+1,\mathbb{C})$-bundle E is fixed for the action of $\bar{\sigma}$ if and only if E lifts to an $\mathrm{SL}(n+1,\mathbb{C})$-bundle fixed by σ.*

Proof. Since the outer involution $\bar{\sigma}$ permutes the connected components via the action of the outer involution of $\mathrm{SL}(n+1,\mathbb{C})$ on its center, it is clear that in the odd case, there are not fixed points. If n is even, $\bar{\sigma}$ restricts to an automorphism of the connected component of $M(\mathrm{PSL}(n+1,\mathbb{C}))$ of those bundles which lift to $M(\mathrm{SL}(n+1,\mathbb{C}))$. Then, Proposition 4.1 concludes the result.

As an immediate consequence of the proposition above, we have the following description of fixed points of $\bar{\sigma}$.

Proposition 4.6. *Let σ be the involution of $M(\mathrm{SL}(n+1,\mathbb{C}))$ induced by the outer involution of A_n and $\bar{\sigma}$ be the corresponding involution of $M(\mathrm{PSL}(n+1,\mathbb{C}))$. Suppose that n is odd. Then,*

$$\mathrm{Fix}(\bar{\sigma}) = \mathrm{Im}\big(\mathrm{Fix}(\sigma) \to M(\mathrm{PSL}(n+1,\mathbb{C}))\big).$$

Moreover, the subvariety of stable $\mathrm{PSL}(n+1,\mathbb{C})$-bundles fixed by $\bar{\sigma}$ is

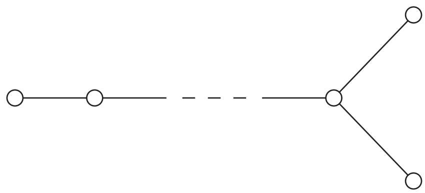
$$\mathrm{Im}\big(N_{\mathrm{s}}(m) \to M(\mathrm{PSL}(n+1,\mathbb{C}))\big),$$

using the notation of Proposition 4.4.

PROOF. The first statement is immediate from Proposition 4.5. For the second, observe that the fixed points in $M(\mathrm{PSL}(n+1, \mathbb{C}))$ for $\bar{\sigma}$ lift to polystable principal $\mathrm{SL}(n+1, \mathbb{C})$-bundles and that, in this situation, a principal $\mathrm{PSL}(n+1, \mathbb{C})$-bundle is stable if and only if the corresponding $\mathrm{SL}(n+1, \mathbb{C})$-bundle is stable.

5. The case of D_n

We will now deal with the case of D_n, $n \geq 4$. Take $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and $G = \mathrm{Spin}(2n, \mathbb{C})$ the simply-connected complex Lie group with Lie algebra $\mathfrak{g}$. The Dynkin diagram of D_n is of type



From the analysis of the group of symmetries of the Dynkin diagram we see that $\mathrm{Out}(\mathfrak{g}) \cong \mathbb{Z}_2$ if $n > 4$ and $\mathrm{Out}(\mathfrak{so}(8, \mathbb{C})) \cong S_3$.

Let $n \geq 4$. Let σ be the nontrivial outer involution of D_n. We consider first the action of σ in $M(\mathrm{Spin}(2n, \mathbb{C}))$.

PROPOSITION 5.1. *Let σ be the involution of $M(\mathrm{Spin}(2n, \mathbb{C}))$ coming from the nontrivial outer involution of D_n. Let E be a polystable principal* $\mathrm{Spin}(2n, \mathbb{C})$*-bundle. Then, E is fixed for the action of σ in $M(\mathrm{Spin}(2n, \mathbb{C}))$ if and only if the orthogonal bundle associated to the* $\mathrm{Spin}(2n, \mathbb{C})$*-bundle E admits a reduction of structure group to the group $S(\mathrm{O}(2r+1, \mathbb{C}) \times \mathrm{O}(2n-2r-1, \mathbb{C}))$, for some r with $0 \leq r \leq n-1$.*

PROOF. If $E \in M(\mathrm{Spin}(2n, \mathbb{C}))$ is fixed by σ, then E admits an automorphism of order 2 and, then, a decomposition $E = E_1 \oplus E_2$ as a direct sum of the eigenspaces corresponding to the eigenvalues 1 and -1; E seen as an orthogonal bundle. Observe that, as σ is an outer involution, the rank of E_1 and the rank of E_2 are odd. Indeed, if $\mathrm{rk}\, E_1$ and $\mathrm{rk}\, E_2$ were even one can easily construct an orthogonal inner automorphism with the same action as σ. Suppose $\mathrm{rk}\, E_1 = 2k$ and $\mathrm{rk}\, E_2 = 2n - 2k$. Take

$$g = \begin{pmatrix} -i_{2k} & \\ & i_{2k} \end{pmatrix}.$$

It is clear that

$$g^2 = \begin{pmatrix} 1_{2k} & \\ & -1_{2k} \end{pmatrix},$$

and it is clear that $\det g = 1$, so $g \in \mathrm{SO}(2n, \mathbb{C})$. Moreover, the inner automorphism given by g equals σ, which is outer. This says that $\operatorname{rk} E_1$ and $\operatorname{rk} E_2$ must be odd. Take $\operatorname{rk} E_1 = 2r + 1$ and $\operatorname{rk} E_2 = 2n - 2r - 1$ for some r with $0 \leq r \leq n - 1$.

On the other hand, the direct sum $E = E_1 \oplus E_2$ is orthogonal for the quadratic form $\langle \cdot\,, \cdot \rangle$ of E coming from its Spin structure. Indeed, if $v_1 \in E_1$ and $v_2 \in E_2$, then

$$\langle v_1, v_2 \rangle = \langle \sigma(v_1), \sigma(v_2) \rangle = -\langle v_1, v_2 \rangle.$$

All this proves that σ is of the form

$$\begin{pmatrix} 1_{2r+1} & \\ & -1_{2n-2r-1} \end{pmatrix},$$

for some r with $0 \leq r \leq n - 1$. It is also easy to see that two such automorphisms for different odd integers r are not conjugate by an inner automorphism.

The bundle E, seen as an orthogonal bundle, admits a reduction of structure group to $S(\mathrm{O}(2r+1, \mathbb{C}) \times \mathrm{O}(2n-2r-1, \mathbb{C}))$, which is of type $B_r \oplus B_{n-r-1}$. This algebra is in fact the subalgebra of fixed points in $\mathfrak{g}$, of type D_n, for the action of σ (see [23, Theorem 5.10]).

Let $n \geq 4$. For each $k \leq 2n$, we have the exact sequence of groups

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(k, \mathbb{C}) \to \mathrm{O}(k, \mathbb{C}) \to 1$$

and, then, the long exact sequence of cohomology groups

$$H^1(\mathrm{Pin}(k, \mathbb{C})) \to H^1(\mathrm{O}(k, \mathbb{C})) \xrightarrow{d_k} \mathbb{Z}_2.$$

Analogously, we have the following

$$\begin{aligned} &H^1(\mathrm{Pin}(2r+1, \mathbb{C}) \times \mathrm{Pin}(2n-2r-1, \mathbb{C})) \\ &\qquad \longrightarrow H^1(\mathrm{O}(2r+1, \mathbb{C}) \times \mathrm{O}(2n-2r-1, \mathbb{C})) \\ &\qquad\qquad \xrightarrow{(d_{2r+1}, d_{2n-2r-1})} \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

It is easy to see that an element of $H^1(\mathrm{O}(2r+1, \mathbb{C}) \times \mathrm{O}(2n-2r-1, \mathbb{C}))$ lifts to an element of $H^1(\mathrm{Spin}(2n, \mathbb{C}))$ if and only if this element annihilates $d_{2r+1} + d_{2n-2r-1}$. We denote

$$\begin{aligned} M_r = \big\{(E_1, E_2) \in M(S(\mathrm{O}(2r+1, \mathbb{C}) \times \mathrm{O}(2n-2r-1, \mathbb{C}))) : \\ d_{2r+1}(E_1) + d_{2n-2r-1}(E_2) = 0\big\}. \end{aligned} \tag{4}$$

It is clear that we have a well-defined morphism $M_r \to M(\mathrm{SO}(2n, \mathbb{C}))$.

DEFINITION 5.2. Suppose that $n \geq 4$. Consider the projection map

$$\pi\colon M(\mathrm{Spin}(2n,\mathbb{C})) \to M(\mathrm{SO}(2n,\mathbb{C})).$$

For $0 \leq r \leq n-1$, we define

$$N'(r) = \pi^{-1}\big(\mathrm{Im}(M_r \to M(\mathrm{SO}(2n,\mathbb{C})))\big),$$

where M_r is the subvariety defined in (4).

Finally, we can rewrite Proposition 5.1 in the following way.

PROPOSITION 5.3. *Let $n \geq 4$. Let σ be the involution of $M(\mathrm{Spin}(2n,\mathbb{C}))$ coming from the outer involution of D_n. Then, the subvariety of fixed points in $M(\mathrm{Spin}(2n,\mathbb{C}))$ for the action of σ is*

$$\bigcup_{r=0}^{n-1} N'(r),$$

where $N'(r)$ is the subvariety of $M(\mathrm{Spin}(2n,\mathbb{C}))$ determined in Definition 5.2.

It is clear that an isotropic subbundle of a fixed point in $M(\mathrm{Spin}(2n,\mathbb{C}))$ is necessarily a direct sum of isotropic subbundles of its decomposition of type $B_r \oplus B_{n-r-1}$. It is then easily seen that a fixed point thus obtained is stable as a Spin-bundle if and only if it can be written as a direct sum of two stable orthogonal bundles of ranks $2r+1$ and $2n-2r-1$ respectively.

For each r as above, we consider M_r the moduli space of principal $S(\mathrm{O}(2r+1,\mathbb{C}) \times \mathrm{O}(2n-2r-1,\mathbb{C}))$-bundles defined in (4). We denote by M_r^{s} the open subvariety of M_r consisting on stable bundles of M_r. Let $M^{\mathrm{s}}(\mathrm{SO}(2n,\mathbb{C}))$ denote the open subset of stable $\mathrm{SO}(2n,\mathbb{C})$-bundles. Then, it is clear that the image of the map

$$\pi_r\colon M_r^{\mathrm{s}} \to M(\mathrm{SO}(2n,\mathbb{C}))$$

is contained in $M^{\mathrm{s}}(\mathrm{SO}(2n,\mathbb{C}))$. This allows us to define the following.

DEFINITION 5.4. Let $n \geq 4$ and $0 \leq r \leq n-1$. Take $M_r = M(S(\mathrm{O}(2r+1,\mathbb{C}) \times \mathrm{O}(2n-2r-1,\mathbb{C})))$, the moduli space of principal $S(\mathrm{O}(2r+1,\mathbb{C}) \times \mathrm{O}(2n-2r-1,\mathbb{C}))$-bundles. Let M_r^{s} be the subvariety of stable bundles of M_r. Consider the forgetful map $\pi_r\colon M_r^{\mathrm{s}} \to M(\mathrm{SO}(2n,\mathbb{C}))$. We define

$$N'_{\mathrm{s}}(r) = \pi^{-1}\big(\mathrm{Im}(M_r^{\mathrm{s}} \to M^{\mathrm{s}}(\mathrm{SO}(2n,\mathbb{C})))\big)$$

and

$$N'_{\mathrm{ps}}(r) = \pi^{-1}\big(\mathrm{Im}(M_r \setminus M_r^{\mathrm{s}} \to M(\mathrm{SO}(2n,\mathbb{C})))\big).$$

Our next result determines the precise relation between fixed points in $M(\mathrm{Spin}(2n, \mathbb{C}))$ of the involution σ and the subvarieties of stable or strictly polystable bundles.

PROPOSITION 5.5. *Let σ be the involution of $M(\mathrm{Spin}(2n, \mathbb{C}))$ coming from the outer involution of D_n, for $n \geq 4$. For $0 \leq r \leq n-1$, let $N'_{\mathrm{s}}(r)$ and $N'_{\mathrm{ps}}(r)$ be the subvarieties determined in Definition 5.4. Then, the subvariety of stable fixed points in $M(\mathrm{Spin}(2n, \mathbb{C}))$ for the action of σ is*

$$\bigcup_{r=0}^{n-1} N'_{\mathrm{s}}(r)$$

and the subvariety of strictly polystable fixed points of σ is

$$\bigcup_{r=0}^{n-1} N'_{\mathrm{ps}}(r).$$

PROOF. It is clear that $N'_{\mathrm{s}}(r)$ consists of stable fixed points for the action of σ, as noted in Section 2.

Every other fixed point thus obtained is strictly polystable seen as a $\mathrm{Spin}(2n, \mathbb{C})$-bundle because it is polystable as an orthogonal bundle. Thus fixed points in $N'_{\mathrm{ps}}(r)$ fall into the nonstable locus of $M(\mathrm{Spin}(2n, \mathbb{C}))$, whenever $g \geq 3$.

We will now determine how the polystable bundles fall into the nonstable locus of the moduli space. In [3], the irreducible components of the nonstable locus of $M(\mathrm{Spin}(2n, \mathbb{C}))$ are described. We briefly sketch this description here.

For each r with $1 \leq r \leq n-1$, let $M(\mathrm{U}(r))$ be the moduli space of polystable unitary bundles of rank r and let J be the Jacobian. Consider the morphisms $\det\colon M(\mathrm{U}(r)) \to J$ and $J \to J$ given by $L \mapsto L^2$. Let $M'(\mathrm{U}(r))$ be their fibre product over J. Then we have a commutative diagram of the form

$$\begin{tikzcd} M'(\mathrm{U}(r)) = M(\mathrm{U}(r)) \times_J J \arrow[r] \arrow[d] & J \arrow[d, "L^2"] \\ J \arrow[r, "\det"] & J \end{tikzcd}$$

We consider the morphism

$$\mathscr{F}_r\colon M'(\mathrm{U}(r)) \times M(\mathrm{Spin}(2n-2r, \mathbb{C})) \to M(\mathrm{Spin}(2n, \mathbb{C}))$$

given by $\mathscr{F}_r(V, F) = H(V) \oplus F$. We denote by $H(V) = V \oplus V^*$ the hyperbolic vector bundle given by V, which comes with a natural quadratic form, thus with an SO-structure. There is an induced Spin-structure on $H(V)$, given by the square root of $\det V$ provided by the projection $M'(\mathrm{U}(r)) \to J$. Denote by $H(r)$ the image of the morphism $\mathscr{F}$, that is,

$$H(r) = \mathrm{Im}\left(M'(\mathrm{U}(r)) \times M(\mathrm{Spin}(2n-2r, \mathbb{C})) \to M(\mathrm{Spin}(2n, \mathbb{C}))\right). \quad (5)$$

For $r = n$, we take the morphism $\mathscr{G}\colon M'(\mathrm{U}(r)) \to M(\mathrm{Spin}(2n, \mathbb{C}))$ given by the map $V \mapsto H(V)$, the Spin-structure of $H(V)$ induced by the Spin structure of V. We then define

$$H(n) = \mathrm{Im}\,\mathscr{G}. \quad (6)$$

In [3], the assertion that there are morphisms as described above is proved and also that the nonstable locus in $M(\mathrm{Spin}(2n, \mathbb{C}))$ is the union of all the subvarieties $H(r)$, $1 \leq r \leq n$,

$$\bigcup_{r=1}^{n} H(r).$$

The nonstable locus is, then, given by n irreducible components.

Proposition 5.6. *Let $n \geq 4$ and σ be the involution of $M(\mathrm{Spin}(2n, \mathbb{C}))$ coming from the outer involution of D_n. Then, the subvariety of strictly polystable fixed points in $M(\mathrm{Spin}(2n, \mathbb{C}))$ for the action of σ is*

$$\bigcup_{r=1}^{n-1} H(r),$$

where, for $1 \leq r \leq n-1$, $H(r)$ is defined in (5).

Proof. Take an element $E \in M(\mathrm{Spin}(2n, \mathbb{C}))$ fixed for σ. Then, there exists r with $0 \leq r \leq n-1$ such that E comes from an element of $M_r \setminus M_r^{\mathrm{s}}$. Then, E is of the form $E = E_1 \oplus E_2$ with E_1 and E_2 orthogonal bundles, $\mathrm{rk}\, E_1 = 2r+1$ and $\mathrm{rk}\, E_2 = 2n-2r-1$. One of the two bundles, E_1 or E_2, or both, are strictly polystable. If E_1 is polystable, there exists k with $1 \leq k \leq r$ such that E_1 is of the form $H(V) \oplus F$ as above with $\mathrm{rk}\, V = k$. Then, it is easily seen that $E \in H(k)$. If E_1 is stable, then E_2 is strictly polystable and the only difference is that k varies from 1 to $n-r-1$. From this, we see that

$$N'_{\mathrm{ps}}(r) \subseteq \bigcup_{k=1}^{\max\{r,n-r-1\}} H(k).$$

Moreover, each element of $\bigcup_{k=1}^{\max\{r,n-r-1\}} H(k)$ of course comes with a reduction of structure group to $S(\mathrm{O}(2r+1,\mathbb{C}) \times \mathrm{O}(2n-2r-2,\mathbb{C}))$, so is fixed by σ. Then, we have the equality

$$N'_{\mathrm{ps}}(r) = \bigcup_{k=1}^{\max\{r,n-r-1\}} H(k).$$

Finally, we have stated that the subvariety of strictly polystable fixed points of σ is

$$\bigcup_{r=0}^{n-1} N'_{\mathrm{ps}}(r) = \bigcup_{r=0}^{n-1} \bigcup_{k=1}^{\max\{r,n-r-1\}} H(k) = \bigcup_{r=1}^{n-1} H(r).$$

The preceding result means that the whole nonstable locus of $M(\mathrm{Spin}(2n,\mathbb{C}))$ is composed of fixed points of σ except for the irreducible component $H(n)$, which has no fixed points. We have then proved the following.

Proposition 5.7. *Let $n \geq 4$ and σ be the involution of $M(\mathrm{Spin}(2n,\mathbb{C}))$ coming from the outer involution of D_n. The $n-1$ irreducible components $H(1),\dots,H(n-1)$ of the strictly polystable locus of $M(\mathrm{Spin}(2n,\mathbb{C}))$ defined in (5) are composed of fixed points of the outer involution σ. Moreover, every nonstable fixed point of σ is in $\cup_{k=1}^{n-2} H(k)$, so the remaining irreducible component of the nonstable locus, $H(n)$ has no fixed points of σ.*

We consider now the group $G = \mathrm{SO}(2n,\mathbb{C})$, $n \geq 4$, whose Lie algebra is $\mathfrak{so}(2n,\mathbb{C})$. There is a short exact sequence of groups

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(2n,\mathbb{C}) \to \mathrm{SO}(2n,\mathbb{C}) \to 1$$

which induces a map $\rho\colon M(\mathrm{Spin}(2n,\mathbb{C})) \to M(\mathrm{SO}(2n,\mathbb{C}))$. The moduli space $M(\mathrm{SO}(2n,\mathbb{C}))$ has two connected components, one of them given by those $\mathrm{SO}(2n,\mathbb{C})$-bundles which lift to $\mathrm{Spin}(2n,\mathbb{C})$-bundles (the image of ρ) and the other given by those which do not lift. The outer involution of $\mathfrak{so}(2n,\mathbb{C})$ gives rise to an outer involution of the group $\mathrm{SO}(2n,\mathbb{C})$, so it induces an involution

$$\bar{\sigma}\colon M(\mathrm{SO}(2n,\mathbb{C})) \to M(\mathrm{SO}(2n,\mathbb{C})).$$

It is easily seen that this involution has no fixed points.

Proposition 5.8. *Let $\bar{\sigma}$ be the outer involution of $M(\mathrm{SO}(2n,\mathbb{C}))$ induced by the outer involution of $\mathfrak{so}(2n,\mathbb{C})$. Then, $\bar{\sigma}$ admits no fixed points.*

Proof. Observe that $\bar{\sigma}$ permutes the two connected components of $M(\mathrm{SO}(2n,\mathbb{C}))$, so it does not leave fixed points.

The analysis for $G = \mathrm{PSO}(2n, \mathbb{C})$ is similar. From the exact sequence of groups

$$1 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Spin}(2n, \mathbb{C}) \to \mathrm{PSO}(2n, \mathbb{C}) \to 1$$

we obtain that $M(\mathrm{PSO}(2n, \mathbb{C}))$ has four connected components and there is a natural map $M(\mathrm{Spin}(2n, \mathbb{C})) \to M(\mathrm{PSO}(2n, \mathbb{C}))$. It is also known that the outer involution of $\mathfrak{so}(2n, \mathbb{C})$ gives rise to an outer involution of $\mathrm{PSO}(2n, \mathbb{C})$, so it induces an involution

$$\bar{\bar{\sigma}} \colon M(\mathrm{PSO}(2n, \mathbb{C})) \to M(\mathrm{PSO}(2n, \mathbb{C})).$$

This involution has no fixed points either.

PROPOSITION 5.9. *Let $\bar{\bar{\sigma}}$ be the outer involution of $M(\mathrm{PSO}(2n, \mathbb{C}))$ induced by the outer involution of $\mathfrak{so}(2n, \mathbb{C})$. Then, $\bar{\bar{\sigma}}$ admits no fixed points.*

PROOF. The outer involution of $\mathrm{Spin}(2n, \mathbb{C})$ acts on its center $\mathbb{Z}_2 \times \mathbb{Z}_2$ by fixing two elements, which correspond exactly to the two connected components of $M(\mathrm{PSO}(2n, \mathbb{C}))$ whose elements correspond to principal bundles which lift to $\mathrm{SO}(2n, \mathbb{C})$-bundles. If E is a polystable $\mathrm{PSO}(2n, \mathbb{C})$-bundle fixed by $\bar{\sigma}$, then its two lifts in $M(\mathrm{SO}(2n, \mathbb{C}))$, E_1 and E_2, differ by an element of $H^1(X, Z(\mathrm{SO}(2n, \mathbb{C})))$.

Now, denote by $\bar{\sigma}$ the outer involution of $M(\mathrm{SO}(2n, \mathbb{C}))$ induced by σ. From Proposition 5.8 we know that $\bar{\sigma}(E_1) \ncong E_1$ and $\bar{\sigma}(E_2) \ncong E_2$. Since $\bar{\bar{\sigma}}(E) \cong E$, it must be $\bar{\sigma}(E_1) \cong E_2$, which contradicts our previous assertion because $\bar{\sigma}$ is induced by an outer involution. Therefore, $\bar{\bar{\sigma}}$ admits no fixed points.

We will now consider the case when $n = 4$ and $G = \mathrm{Spin}(8, \mathbb{C})$. We have that $\mathrm{Out}(D_4) \cong S_3$, so, in addition to the outer involution σ, there is an outer automorphism of order three, τ, of $\mathrm{Spin}(8, \mathbb{C})$, called the triality automorphism. The only possibilities for the group of fixed points of such automorphism are G_2 or $\mathrm{PSL}(3, \mathbb{C})$ (see [23]). The following result is proved in [2, Theorem 7.2].

PROPOSITION 5.10. *The subvariety of fixed points of the triality automorphism τ in $M(\mathrm{Spin}(8, \mathbb{C}))$ is*

$$\mathrm{Im}\big(M(G_2) \to M(\mathrm{Spin}(8, \mathbb{C}))\big) \cup \mathrm{Im}\big(M(\mathrm{PSL}(3, \mathbb{C})) \to M(\mathrm{Spin}(8, \mathbb{C}))\big).$$

These fixed points described above are in fact strictly polystable bundles of $M(\mathrm{Spin}(8, \mathbb{C}))$. This is proved in the following Proposition ([3, Theorem 7.2]):

PROPOSITION 5.11. *The subvariety of fixed points in $M(\mathrm{Spin}(8, \mathbb{C}))$ for the action of the triality automorphism τ is contained in the strictly polystable* *locus of* $M(\mathrm{Spin}(8,\mathbb{C}))$. *Moreover, it is contained in the irreducible component* $H(3)$ *of* $M(\mathrm{Spin}(8,\mathbb{C}))$, *defined in* (5).

We briefly sketch the idea of the proof in order to illustrate how the principal G_2-bundles can be seen as $\mathrm{Spin}(8,\mathbb{C})$-bundles. The group G_2 has two irreducible representations, called the fundamental representations. These are the adjoint representation, of dimension 14, and its action on the imaginary octonions, of dimension 7. The last representation is an orthogonal representation

$$\rho\colon G_2 \to \mathrm{SO}(7,\mathbb{C}).$$

Via this representation, G_2 can be seen as the group of automorphisms of $\mathbb{C}^7$ which preserve a non-degenerate 3-form (see [6]).

The triality automorphism induces a decomposition of the Lie algebra $\mathfrak{so}(8,\mathbb{C})$ into vector subspaces of the form

$$\mathfrak{so}(8,\mathbb{C}) = \mathfrak{g}_2 \oplus \mathfrak{h}_+ \oplus \mathfrak{h}_-,$$

where $\mathfrak{h}_+$ and $\mathfrak{h}_-$ are the eigenspaces with eigenvalues $e^{i2\pi/3}$ and $e^{-i2\pi/3}$, respectively. These subspaces are both of complex dimension 7 and the restriction of the triality automorphism gives rise to representations of $\mathfrak{g}_2$ in them, which are mutually dual. The representations induce the two fundamental representations of G_2 explained above. Now, the inclusion $G_2 \to \mathrm{SO}(8,\mathbb{C})$ admits a factorization through the fundamental 7-dimensional representation of G_2, which is also an orthogonal representation.

$$G_2 \to \mathrm{SO}(7,\mathbb{C}) \hookrightarrow \mathrm{SO}(7,\mathbb{C}) \oplus \mathbb{C}^* \hookrightarrow \mathrm{SO}(8,\mathbb{C}).$$

Suppose that $E \in M(\mathrm{Spin}(8,\mathbb{C}))$ admits a reduction of structure group to G_2. Let F be this G_2-bundle. Then, there exists a line bundle L such that the induced $\mathrm{SO}(8,\mathbb{C})$-bundle is $E = F \oplus L$. Moreover, the fundamental 7-dimensional representation of G_2 is of the form $W \oplus W^* \oplus \mathbb{C}$ (see [8, Chapter 22]) with W of rank 3. This says that E is always polystable and $E \in H(3)$, with the notation of (5).

In [3] it is also proved that all the bundles in $H(3)$ are fixed by the action of the triality automorphism, τ. This concludes that the subvariety of fixed points of τ is $H(3)$ ([3, Proposition 7.5]).

PROPOSITION 5.12. *Let* $H(3)$ *be the subvariety of* $M(\mathrm{Spin}(8,\mathbb{C}))$ *defined in* (5). *Take* $E \in H(3)$. *Then,* $\tau(E) = E$, *where* τ *denotes the action of the triality automorphism in* $M(\mathrm{Spin}(8,\mathbb{C}))$.

Finally, we have the following proposition, which is not explicitly proved in [3].

PROPOSITION 5.13. *Let σ be the involution of $M(\mathrm{Spin}(8,\mathbb{C}))$ induced by the outer involution of D_4 and τ be the action on $M(\mathrm{Spin}(8,\mathbb{C}))$ of the triality automorphism. Let $M^\sigma(\mathrm{Spin}(8,\mathbb{C}))$ be subvariety of fixed points in $M(\mathrm{Spin}(8,\mathbb{C}))$ for the action of σ, $M^\sigma_{\mathrm{ps}}(\mathrm{Spin}(8,\mathbb{C}))$ be the subset of strictly polystable fixed points and $M^\tau(\mathrm{Spin}(8,\mathbb{C}))$ be the subvariety of fixed points of τ. Then,*

$$M^\tau(\mathrm{Spin}(8,\mathbb{C})) \subsetneq M^\sigma_{\mathrm{ps}}(\mathrm{Spin}(8,\mathbb{C})) \subsetneq M^\sigma(\mathrm{Spin}(8,\mathbb{C})).$$

In particular, $M^\tau(\mathrm{Spin}(8,\mathbb{C}))$ is contained in the strictly polystable locus of $M(\mathrm{Spin}(8,\mathbb{C}))$.

PROOF. The nonstable locus of $M(\mathrm{Spin}(8,\mathbb{C}))$ is composed of four irreducible components defined in (5) and (6): $H(1)$, $H(2)$, $H(3)$ and $H(4)$. From Proposition 5.7 we know that the subset of strictly polystable fixed points of the outer involution σ is exactly $H(1)\cup H(2)\cup H(3)$. And the whole subvariety of fixed points for the action of τ is exactly $H(3)$, by the preceding proposition. Then, we have that

$$\begin{aligned} M^\tau(\mathrm{Spin}(8,\mathbb{C})) &= H(3) \\ &\subsetneq H(1)\cup H(2)\cup H(3) \\ &= M^\sigma_{\mathrm{ps}}(\mathrm{Spin}(8,\mathbb{C})) \\ &\subsetneq M^\sigma(\mathrm{Spin}(8,\mathbb{C})). \end{aligned}$$

The result is then proved.

Consider now $G = \mathrm{PSO}(8,\mathbb{C})$, the centerless complex simple Lie group whose Lie algebra is also of type D_4 (the other complex simple Lie group with this Lie algebra is $\mathrm{SO}(8,\mathbb{C})$, but observe that the triality automorphism does not descend to an automorphism of $\mathrm{SO}(8,\mathbb{C})$). The group $\mathrm{Spin}(8,\mathbb{C})$ covers the group $\mathrm{PSO}(8,\mathbb{C})$ in this way:

$$1 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Spin}(8,\mathbb{C}) \to \mathrm{PSO}(8,\mathbb{C}) \to 1$$

and there is a map $\rho\colon M(\mathrm{Spin}(8,\mathbb{C})) \to M(\mathrm{PSO}(8,\mathbb{C}))$. The moduli space $M(\mathrm{PSO}(8,\mathbb{C}))$ has four connected components, only one of them given by bundles which lift to $M(\mathrm{Spin}(8,\mathbb{C}))$.

The triality automorphism gives rise to an automorphism of order three of $M(\mathrm{PSO}(8,\mathbb{C}))$

$$\bar{\tau}\colon M(\mathrm{PSO}(8,\mathbb{C})) \to M(\mathrm{PSO}(8,\mathbb{C})).$$

In the next result we study fixed points for this automorphism.

Proposition 5.14. *Let $\bar{\tau}$ be the automorphism of order three of* $M(\mathrm{PSO}(8, \mathbb{C}))$ *induced by the triality automorphism. Then, the subvariety of fixed points of $\bar{\tau}$ is*

$$\mathrm{Fix}(\bar{\tau}) = \mathrm{Im}(\mathrm{Fix}(\tau) \to M(\mathrm{PSO}(8, \mathbb{C}))),$$

where τ denotes also the action of the triality automorphism in $M(\mathrm{Spin}(8, \mathbb{C}))$.

Proof. Observe that $\bar{\tau}$ permutes the four connected components of $M(\mathrm{PSO}(8, \mathbb{C}))$ leaving invariant only one of them, which is the image of the map $\rho\colon M(\mathrm{Spin}(8, \mathbb{C})) \to M(\mathrm{PSO}(8, \mathbb{C}))$. Then, any fixed point for $\bar{\tau}$ is the image by ρ of a $\mathrm{Spin}(8, \mathbb{C})$-bundle which is fixed for τ.

6. Albert algebras and exceptional groups

The classification of simple complex Lie groups consists of four infinite series, the classical Lie groups considered above, and five exceptional complex Lie groups, called G_2, F_4, E_6, E_7, and E_8 (a good review about exceptional groups can be read in [1]). We are interested in E_6, the only exceptional simple complex Lie group which admits nontrivial outer automorphisms, and F_4, which appears as the subgroup of fixed points of these automorphisms.

For $n \geq 2$, we consider the algebra of $n \times n$ complex matrices equipped with the natural matrix product. It is well-known that this is an associative and non-commutative algebra. From this product we can derive a commutative product $\circ$, called the Jordan product, by defining

$$A \circ B = \tfrac{1}{2}(AB + BA).$$

This Jordan product is non-associative but it satisfies the so called Jordan identity, which establishes that

$$((A \circ A) \circ B) \circ A = (A \circ A) \circ (B \circ A)$$

for all A and B. From this product, a symmetric bilinear form can be defined by taking the trace: $\mathrm{Tr}(A \circ B)$. A complex Jordan algebra is then defined to be the non-associative commutative algebra given by the Jordan product $\circ$, which satisfies the Jordan identity.

Simple Jordan algebras over $\mathbb{C}$ are completely classified. It turns out that apart from those simple Jordan algebras which arise from associative matrix algebras over $\mathbb{C}$ as we have described, there is only one more simple Jordan algebra, sometimes called the exceptional Jordan algebra or the Albert algebra ([22, Sec. 5.8]). It may be constructed by introducing the Jordan product in the algebra $\mathfrak{h}_3(\mathbb{O})$ of 3×3 Hermitian matrices over the algebra of the octonions, that is, matrices of the form

$$\begin{pmatrix} p & C & \overline{B} \\ \overline{C} & q & A \\ B & \overline{A} & r \end{pmatrix},$$

where $p, q, r \in \mathbb{R}$. It is clear that the Albert algebra so defined has dimension 27.

The Jordan product gives rise to three forms on the Albert algebra: a linear form a, a bilinear form b and a trilinear form c. These forms are defined as follows.

$$\begin{aligned} a(x) &= \mathrm{Tr}(x), \\ b(x, y) &= \mathrm{Tr}(x \circ y), \\ c(x, y, z) &= \mathrm{Tr}((x \circ y) \circ z). \end{aligned}$$

The Jordan product is commutative, so the bilinear form b is symmetric and then induces a quadratic form B. From the Jordan identity it can be seen that $c(x, y, z) = c(y, z, x)$, so that the trilinear form c is also symmetric. It then gives rise to the so called norm N.

For $x \in \mathfrak{h}_3(\mathbb{O})$, it can be seen that the minimal polynomial, m_x, of x is of the form

$$m_x(t) = t^3 - a(x)t^2 + B(x)t - N(x).$$

We may define E_6 to be the group of linear automorphisms of the complexification of the Albert algebra which preserves the norm N. That is,

$$E_6 = \{\alpha \in \mathrm{GL}(\mathfrak{h}_3(\mathbb{O}))^{\mathbb{C}} : N(\alpha(x)) = N(x)\}.$$

The group E_6 so defined is the simply-connected complex Lie group with Lie algebra $\mathfrak{e}_6$.

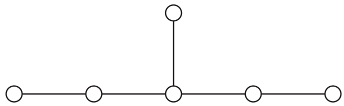
We can also define the Lie group F_4 as the group of automorphisms of the Albert algebra. These automorphisms are precisely those linear maps which preserves the minimal polynomial, so F_4 contains all linear automorphisms of $\mathfrak{h}_3(\mathbb{O})^{\mathbb{C}}$ which preserve the three forms a, B and N. It is then clear that F_4 is a subgroup of E_6.

The exceptional algebra $\mathfrak{f}_4$ is then the algebra of derivations of the Albert algebra.

The Albert algebra gives rise to a 27-dimensional complex representation of E_6, $E_6 \to \mathrm{SL}(\mathfrak{h}_3(\mathbb{O})^{\mathbb{C}})$. This is a fundamental representation. This representation is irreducible and faithful and it is inequivalent to its dual, which of course is also 27-dimensional irreducible and faithful. In fact, these are the smallest irreducible representations of E_6.

The identity matrix in $\mathfrak{h}_3(\mathbb{O})^{\mathbb{C}}$ acts as an identity element of the Albert algebra and its orthogonal complement for the bilinear form b consists on the 26-dimensional subspace of matrices of trace 0. The restriction of the fundamental 27-dimensional representation of E_6, explained above, to F_4 gives rise to the fundamental 26-dimensional representation of F_4.

The Dynkin diagram of E_6 is of the form



Then, there exists only one nontrivial outer automorphism of E_6, σ, which corresponds to the unique nontrivial symmetry of the Dynkin diagram. This symmetry has order 2, so σ is an involution. In fact, no other exceptional simple Lie group admits nontrivial outer automorphisms.

It is well-known that there exists a natural correspondence between nodes of the Dynkin diagram and the fundamental representations of the simple Lie algebra. In the case of E_6, the fundamental representations have dimensions 27, 351, 2925, 351, 27 and 78, corresponding to the nodes of the Dynkin diagram read in the five-node chain first, with the last node being connected to the middle one. The two fundamental 27-dimensional representations are related by the outer involution σ, so σ acts by taking the dual of the vector representation.

We now describe briefly the parabolic subgroups of E_6 in order to give later an appropriate notion of stability for E_6-bundles (a complete description of parabolic subgroups of E_6 and the induced filtrations can be read in [20], following [15]). The group E_6 is the group of automorphisms of a 27-dimensional complex vector space V equipped with a holomorphic 3-form Ω. Recall that an isotropic subspace of V is a subspace W such that $\Omega(W, W, W) = 0$. It is said to be maximal isotropic if it is not properly contained in other isotropic subspace. The vector space V can be written as a direct sum of the form

$$V = \mathbb{C}^6 \oplus \mathbb{C}^6 \oplus \bigwedge^2 (\mathbb{C}^6)^*.$$

This is called the Cartan decomposition, introduced by Cartan in his thesis [7] of 1894. In terms of this decomposition, one can define a cubic form D on V of the form $D(v) = \mathrm{Pf}(x) + \langle z, x \wedge y \rangle$, where Pf denotes the Pfaffian and $v = (x, y, z) \in V$. The trilinear form Ω then comes from the cubic form D. The group E_6 admits exactly 6 parabolic subgroups, which give filtrations of the form

$$0 \subsetneq V_r \subsetneq V,$$

where

$$V_r = \mathbb{C}^6 \oplus \mathbb{C}^{6-r} \oplus \bigwedge^2 (\mathbb{C}^r)^* \tag{7}$$

is clearly an isotropic subspace, for $r = 0, \dots, 5$, $r \neq 1$ (since for $v = (x, y, z) \in V_r$, the Pfaffian of z is zero and the term $\langle z, x \wedge y \rangle$ is necessarily zero by construction of V_r), and

$$0 \subsetneq V_1 \subsetneq V_0 \subsetneq V$$

for $r = 1$. All of them are maximal isotropic except for $r = 1$. In this case, V_0 is an isotropic subspace and $V_1 \subsetneq V_0$. These subspaces have dimensions 12, 11, 11, 12, 14 and 17. The weights for these filtrations are triples of real numbers

$$\lambda_1 < \lambda_2 < \lambda_3$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 0$ when the parabolic subgroup is not maximal; $\lambda_1 = \lambda_2 < \lambda_3$ with $2\lambda_1 + \lambda_3 = 0$ when the parabolic is maximal. To see this, observe that the condition for the group E_6 of being invariant under Ω passes to the Lie algebra $\mathfrak{e}_6$ by polarizing and then, for $s \in \mathfrak{e}_6$ and $x, y, z \in V$, we have

$$\Omega(s(x), y, z) + \Omega(x, s(y), z) + \Omega(x, y, s(z)) = 0.$$

If we diagonalize s and take x, y, z to be eigenvectors with eigenvalues λ_x, λ_y, λ_z, then, by linearity,

$$(\lambda_x + \lambda_y + \lambda_z)\Omega(x, y, z) = 0.$$

When the condition $\Omega(x, y, z) \neq 0$ holds, we have $\lambda_x + \lambda_y + \lambda_z = 0$. We call these weights λ_1, λ_2, λ_3. It is easily seen that in the case when the parabolic subgroup is maximal, with induced maximal isotropic subspace V_r, it must be $\lambda_1 = \lambda_2$ and there exists $z \in V$ and $x, y \in V_r$ nonzero vectors such that $\Omega(x, y, z) \neq 0$ (in other case, V_r should not be maximal isotropic). Then, $2\lambda_1 + \lambda_3 = 0$.

Finally, recall that there is another complex simple group with Lie algebra $\mathfrak{e}_6$ which we will call $\bar{E}_6$. This is a centerless group whose universal covering is E_6. We have a short exact sequence of the form

$$1 \to \mathbb{Z}_3 \to E_6 \to \bar{E}_6 \to 1. \tag{8}$$

7. Principal E_6-bundles

In terms of the fundamental 27-dimensional representation of E_6, a principal E_6-bundle can be seen as a complex vector bundle with rank 27 and trivial determinant equipped with a global holomorphic symmetric non-degenerate 3-form Ω. Given a principal E_6-bundle E, a subbundle F of E is said to be isotropic if $\Omega(F, F, F) = 0$. The isotropic subbundle F is said to be maximal isotropic if it is not properly contained in other isotropic subbundle.

Let E be a principal E_6-bundle. It follows from the discussion of parabolic subgroups of E_6 at the end of the previous section that any filtration of E induced by a parabolic subgroup is of the form

$$0 \subsetneq E_r \subsetneq E,$$

if the parabolic subgroup is maximal, where E_r is an isotropic subbundle of E of dimension 12, 11, 12, 14 or 17, depending on $r = 0, \dots, 5$, $r \neq 1$ in (7), and the filtration is of the form

$$0 \subsetneq E_1 \subsetneq E_0 \subsetneq E,$$

where E_0 is isotropic and $\mathrm{rk}\, E_1 = 11$, if the parabolic subgroup is not maximal. Observe that, when $r \neq 1$, E_r is always isotropic. A reduction of E to a Levi subgroup of a maximal parabolic subgroup gives rise to a decomposition of E into a direct sum of vector subbundles of the form

$$E = F \oplus (E/F).$$

If the parabolic subgroup is not maximal (the only possibility is $r = 1$), the decomposition is

$$E = E_1 \oplus (E_0/E_1) \oplus (E/E_0).$$

The degree of a filtration given by a parabolic subgroup of E_6 is

$$(\lambda_1 - \lambda_3) \deg E_r,$$

for some weights $\lambda_1 < \lambda_3$ with $2\lambda_1 + \lambda_3 = 0$ if the parabolic subgroup is maximal (that is, if $r \neq 1$), and

$$(\lambda_1 - \lambda_2) \deg E_1 + (\lambda_2 - \lambda_3) \deg E_0,$$

if the parabolic is not maximal. Since the semistability condition says that this degree must be ≥ 0 (> 0 for stability), for every choice of maximal parabolic subgroup and weights (see Section 2), and E_r is isotropic for maximal parabolic subgroups, we obtain the following notion of stability for E_6-bundles.

DEFINITION 7.1. Let E be a principal E_6 bundle. Then E is *semistable* if for every isotropic subbundle F of E we have $\deg F \leq 0$. It is *stable* if for every isotropic subbundle F of E we have $\deg F < 0$. It is *polystable* if it can be written as a direct sum of stable vector subbundles

$$E = F \oplus (E/F),$$

where F is isotropic and $\deg F = 0$, or it can be written as a direct sum of stable vector subbundles

$$E = E_1 \oplus (E_0/E_1) \oplus (E/E_0),$$

where E_0 is isotropic, $E_1 \subsetneq E_0$ and $\deg E_1 = \deg E_0 = 0$.

In this section, we consider the moduli space $M(E_6)$ of polystable principal E_6-bundles. This moduli space is an algebraic variety of dimension $78(g-1)$ and it is irreducible, because the structure group E_6 is simply connected.

Let σ be the non-trivial outer involution of E_6. Then, σ acts on the moduli space of principal E_6-bundles inducing an automorphism of order 2 of the moduli. If we view E as a vector bundle of rank 27, then σ acts by taking the dual bundle, that is, $\sigma(E) = E^*$. This follows from the fact that σ acts interchanging the fundamental 27-dimensional representation and its dual.

In [23, Theorem 5.10] it is proved that the outer involution σ of E_6 has two different lifts via the morphism $\mathrm{Aut}(E_6) \to \mathrm{Out}(E_6)$, whose subgroups of fixed points are F_4 and $\mathrm{Sp}(8, \mathbb{C})$. Our goal is to see that fixed points in $M(E_6)$ for the induced action of σ are exactly E_6-bundles which admit a reduction of the structure group to F_4 or $\mathrm{Sp}(8, \mathbb{C})$. The group $\mathrm{Sp}(8, \mathbb{C})$ is a subgroup of F_4, so every fixed point will reduce its structure group to F_4.

We begin by proving that if an E_6-bundle in $M(E_6)$ is fixed for σ, then it is strictly polystable. Then, we will study strictly polystable fixed points.

PROPOSITION 7.2. *Let σ be the involution of $M(E_6)$ coming from the outer involution of $\mathfrak{e}_6$. Let $E \in M(E_6)$ be a fixed point for the action of σ. Then E is strictly polystable.*

PROOF. The bundle E is fixed for σ, so there exists an isomorphism $f\colon E \to E^*$. This isomorphism f defines an automorphism $\alpha = (f^{-1})^t f\colon E \to E$ of order 2. Suppose first that $\alpha \neq \mathrm{id}$. We know that the subbundles

$$F = \ker(\alpha - \mathrm{id}) \quad \text{and} \quad L = \ker(\alpha + \mathrm{id}) \tag{9}$$

are proper subbundles of E and satisfy that L is an isotropic subbundle and that E is an extension of the form

$$0 \to F \to E \to L \to 0.$$

Taking the dual, we have

$$0 \to L^* \to E^* \cong E,$$

because E is a fixed point for σ, so L^* is also an isotropic subbundle of E. Then, we have found a proper isotropic subbundle of E (L or L^*) with degree greater or equal to 0. This proves that E is not stable.

Suppose now that $\alpha = \mathrm{id}$. Then, $f = f^t$ defines a nondegenerate symmetric bilinear form on E, which induces a reduction of structure group of E to F_4. Call this reduction F. Since the representation of F_4 given by $F_4 \hookrightarrow E_6 \to \mathrm{GL}(27, \mathbb{C})$ factors through the fundamental 26-dimensional representation of F_4, the principal E_6-bundle E must be strictly polystable.

As in the case of classical simple groups (except for some fixed points in groups of type A_n), fixed points in $M(E_6)$ for the action of the outer involution σ are strictly polystable. We shall prove that these fixed points always admit a reduction of structure group to F_4. This follows from studying the action of σ on a convenient reduction of the bundle to the Levi subgroup of a parabolic subgroup of E_6.

Proposition 7.3. *Let σ be the involution of $M(E_6)$ coming from the outer involution of $\mathfrak{e}_6$. Let E be a strictly polystable principal E_6-bundle, fixed by σ. Then, E admits a reduction of structure group to F_4.*

Proof. We start from a strictly polystable E_6-bundle E fixed by σ. Then, there exists an isomorphism $f\colon E \to E^*$. Since E is strictly polystable, it admits a reduction of structure group, F, to the Levi subgroup L of a parabolic subgroup of E_6 (the Jordan-Hölder reduction). Then, F can be written as a direct sum of stable vector subbundles,

$$F = F_1 \oplus \cdots \oplus F_r.$$

Since every F_i is stable as a vector bundle, the isomorphism f maps F_i to some $F^*_{\alpha(i)}$ for some permutation α of the set $\{1, \ldots, r\}$. By composing f with the isomorphism induced by the permutation α, we may suppose that f restricts to give isomorphisms $f_i\colon F_i \to F_i^*$ for each i. For $i = 1, \ldots, r$ we consider

$$g_i = \left(f_i^t\right)^{-1} f_i\colon F_i \to F_i.$$

Since F_i is stable as a vector bundle, it is simple, so g_i is the multiplication by a scalar $\lambda \in \mathbb{C}^*$. The scalar λ is then an eigenvalue of $g = (f^t)^{-1} f$, which is an automorphism of E as a principal E_6-bundle. Every eigenvalue of g must be of order three because the automorphism leaves invariant the 3-form of E.

Then, $\lambda^3 = 1$. On the other hand, by transposing g_i, one obtains $f_i^t = \lambda f_i$, so $f_i = \lambda^2 f_i^t$. This means that $\lambda^2 = \lambda$. Therefore, $\lambda = 1$.

This proves that $g_i = \mathrm{id}$ for all i, so $g = \mathrm{id}$. Therefore, f induces an orthogonal structure on E and then a reduction of structure group F of E to the intersection $H = \mathrm{SO}(27, \mathbb{C}) \cap E_6$. If $e \in F$ it must satisfy $f(eg) = f(e)g$ for all $g \in H$, because the orthogonal structure is defined by f (in fact, H is defined by this condition). Now, if $g \in E_6$ is fixed by the outer involution of E_6 (we also call it σ) then, by definition of $\sigma(E)$,

$$f(eg) = f(e)\sigma(g) = f(e)g$$

for all $e \in F$. Moreover, since the action of E_6 is faithful, this equality identifies the elements of the subgroup of E_6 of fixed points of σ, which is isomorphic to F_4. Then, E admits a reduction of structure group to F_4.

REMARK 7.4. It is easily seen that every strictly polystable E_6-bundle admits automorphisms not coming from the center of E_6 (it is clear from the Jordan-Hölder reduction of the bundle and the structure of the Levi subgroups of E_6). Since every fixed point of the outer involution σ of $M(E_6)$ is strictly polystable (from Proposition 7.2), we see that an E_6-bundle which is fixed by σ is not simple.

By combining Propositions 7.2 and 7.3, we have finally seen that fixed points in $M(E_6)$ for the action of σ are strictly polystable and admit a reduction of structure group to F_4. One can easily see that the converse is of course true, so we finally establish the following, which is the main result of the section.

THEOREM 7.5. *Let σ be the involution of $M(E_6)$ induced by the outer involution of $\mathfrak{e}_6$. The subvariety of fixed points in $M(E_6)$ for the action of σ lies in the nonstable locus of $M(E_6)$ and can be described as*

$$\mathrm{Im}\big(M(F_4) \to M(E_6)\big).$$

PROOF. If E is a principal E_6-bundle which admits a reduction of structure group $r\colon F \hookrightarrow E$ to the subgroup $\mathrm{Fix}(\sigma)$ (isomorphic to F_4), then $\sigma(r)\colon \sigma(F) = F \to \sigma(E)$ is a reduction of structure group of $\sigma(E)$ to $\sigma(\mathrm{Fix}(\sigma)) = \mathrm{Fix}(\sigma)$. Now it is clear that

$$E \cong F \times_{\mathrm{Fix}(\sigma)} E_6,$$

so

$$\sigma(E) \cong \sigma(F) \times_{\mathrm{Fix}(\sigma)} E_6 \cong F \times_{\mathrm{Fix}(\sigma)} E_6.$$

Thus we have an isomorphism $E \cong \sigma(E)$.

The converse is a consequence of Propositions 7.2 and 7.3.

We will now consider the case $G = \bar{E}_6$ defined in (8), which is the centerless complex simple Lie group whose Lie algebra is $\mathfrak{e}_6$. From the exact sequence (8), it follows that there exists a natural map $\rho\colon M(E_6) \to M(\bar{E}_6)$ and that $M(\bar{E}_6)$ has three connected components, exactly one of them being the image of ρ. It is also easy to see that the outer involution of $\mathfrak{e}_6$ gives rise to an involution

$$\bar{\sigma}\colon M(\bar{E}_6) \to M(\bar{E}_6).$$

PROPOSITION 7.6. *Let σ be the involution of $M(E_6)$ induced by the outer involution of $\mathfrak{e}_6$. Let $\bar{\sigma}$ be the involution of $M(\bar{E}_6)$ defined above. Then, the subvariety of fixed points of $\bar{\sigma}$ in $M(\bar{E}_6)$ is*

$$\mathrm{Fix}(\bar{\sigma}) = \mathrm{Im}\big(\mathrm{Fix}(\sigma) \to M(\bar{E}_6)\big).$$

PROOF. It is clear that the involution $\bar{\sigma}$ acts by permuting two of the connected components of $M(\bar{E}_6)$ and leaving invariant $\mathrm{Im}(\rho)$, defined above. Then, a fixed point of $\bar{\sigma}$ in $M(\bar{E}_6)$ necessarily lifts to a fixed point of σ in $M(E_6)$ and the result holds.

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