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LOW REGULARITY FUNCTION SPACES OF N-VALUED MAPS ARE CONTRACTIBLE

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Abstract

Let M be a compact Lipschitz submanifold, possibly with boundary, of $\mathbb{R}^n$. Let $N \subset \mathbb{R}^k$ be an arbitrary set. Let $s \geq 0$ and $1 \leq p < \infty$ be such that $sp < 1$. Then $W^{s,p}(M; N)$ is contractible.

1. Introduction

This note is motivated by the following result from [6]: if N is a compact manifold, then $L^2(\mathbb{S}^1; N)$ is a contractible space.

We prove the following.

Theorem 1. *Let M be a compact Lipschitz submanifold, possibly with boundary, of $\mathbb{R}^n$. Let $N \subset \mathbb{R}^k$ be an arbitrary set. Let $s \geq 0$ and $1 \leq p < \infty$ be such that $sp < 1$. Then the space*

$$W^{s,p}(M; N) := \{u \in W^{s,p}(M; \mathbb{R}^k) : u(x) \in N \textit{ for a.e. } x \in M\}$$

is contractible.

This contains as special cases the result in [6] and also the fact that, when $\Omega \subset \mathbb{R}^n$ is a smooth bounded open set and $sp < 1$, the space $W^{s,p}(\Omega; \mathbb{S}^1)$ is path-connected [3]. Note that in particular the theorem applies when M is (the closure of) a Lipschitz bounded open set in $\mathbb{R}^n$.

When $sp \geq 1$, the conclusion of the theorem does not hold in general, since maps in $W^{s,p}(\mathbb{S}^1; \mathbb{S}^1)$ have a non trivial topological invariant, the winding number [5]. However, for some M's and N's one may expect the conclusion of Theorem 1 to hold. Here is such a special case, asked by A. Bahri [2]: let B be the unit ball in $\mathbb{R}^n$, with $n \geq 2$. Is it true that $H^{1/2}(B; \mathbb{S}^1)$ is contractible? This question is open.

Let us note that an alternative to $W^{s,p}(M; N)$ is to consider the closure $Z^{s,p}(M; N)$ of $C^\infty(M; N)$ for the $W^{s,p}$ norm. When N is a smooth compact connected manifold and $sp < 1$, we have $Z^{s,p}(M; N) = W^{s,p}(M; N)$. Indeed,

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when $s > 0$ and $M = \overline{\Omega}$, with $\Omega \subset \mathbb{R}^n$ smooth bounded open set, this was obtained in [4]. The case $s = 0$ is obtained by approximating maps f in $L^p(M; N)$ with step functions $g = \sum a_j \mathbb{1}_{A_j}$, where the A_j's are Lipschitz open subsets of M. Such a g belongs to $W^{s,p}(M; N)$ whenever $sp < 1$, and therefore it can be approximated in $W^{s,p}$ (and thus in L^p) with smooth N-valued maps. The above can be extended to Lipschitz manifolds, but we do not pursue this route here.

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2. Proof of Theorem 1

We start by describing the functional setting. Let ℓ be the dimension of M. Then

$$W^{s,p}(M; \mathbb{R}^k) := \left\{ f : M \to \mathbb{R}^k : \|f\|_{W^{s,p}} := \|f\|_{L^p(M)} + |f|_{W^{s,p}(M)} < \infty \right\},$$

where $|\cdot|_{W^{s,p}(M)}$ stands for the Gagliardo type semi-norm

$$|f|_{W^{s,p}(M)} := \left(\int_M \int_M \frac{|f(x) - f(y)|^p}{|x - y|^{\ell+sp}} \, dx \, dy \right)^{1/p}.$$

The norm (respectively the semi-norm) $\|\cdot\|_{W^{s,p}(U)}$ (respectively $|\cdot|_{W^{s,p}(U)}$) on a Lipschitz open subset of M are defined similarly.

PROOF OF THEOREM 1. In Step 1, we prove the result when $M = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is a Lipschitz bounded set. The argument in this special case contains the main idea of the proof in the general case. In Step 2, we present the proof in the general case.

Step 1. Proof when M is the closure of a Lipschitz bounded open set $\Omega \subset \mathbb{R}^n$.

Fix a point $P \in N$. With no loss of generality, we may assume that $P = 0$. Consider the operator T defined by

$$W^{s,p}(M; Ne) \ni u \mapsto Tu\colon \mathbb{R}^n \to N, \quad Tu = \begin{cases} u, & \text{in } M, \\ 0, & \text{in } \mathbb{R}^n \setminus M. \end{cases}$$

By Lemma 2 below, T maps $W^{s,p}(M; N)$ into $W^{s,p}(\mathbb{R}^n; N)$ continuously.

LEMMA 2. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded open set in $\mathbb{R}^n$. Let $s \geq 0$ and $1 \leq p < \infty$ be such that $sp < 1$. Then the extension operator*

$$W^{s,p}(\Omega; \mathbb{R}) \ni u \mapsto v = \begin{cases} u, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

is continuous from $W^{s,p}(\Omega; \mathbb{R})$ into $W^{s,p}(\mathbb{R}^n; \mathbb{R})$.

Granted Lemma 2, we complete Step 1 as follows. Let $R\colon W^{s,p}(\mathbb{R}^n) \to W^{s,p}(M)$ be the restriction operator. Fix a vector $e \in \mathbb{R}^n$ with $|e| > \operatorname{diam} M$, and define the continuous map

$$W^{s,p}(M; N) \times \mathbb{R} \ni (u, t) \mapsto H(u, t) := R[(Tu)(\cdot - te)] \in W^{s,p}(M; N).$$

Then $H(\cdot, 0) = \text{Id}$ and $H(\cdot, 1) = 0$. Thus $W^{s,p}(M; N)$ is contractible.

Step 2. Proof in the general case.

Again with no loss of generality, we may assume that $0 \in N$. The main idea is to "localize" the argument in Step 1 in order to prove the following key fact.

LEMMA 3. *Let $s \geq 0$ and $1 \leq p < \infty$ be such that $sp < 1$. For every point $x \in M$, there exist a neighborhood $x \in U_x \subset M$ and a continuous mapping H_x,*

$$W^{s,p}(M; N) \times [0, 1] \ni (u, t) \mapsto H_x(u, t) \in W^{s,p}(M; N),$$

such that

(1) $H_x(u, 0) = u, \forall u \in W^{s,p}(M; N)$,
(2) $H_x(u, 1) = 0$ *in* $U_x, \forall u \in W^{s,p}(M; N)$,
(3) $H_x(u, t) = u$ *in* $M \setminus U_x, \forall u \in W^{s,p}(M; N), \forall t \in [0, 1]$.

Granted Lemma 3, we argue as follows. Cover M with a finite number of neighborhoods $U_{x_1}, \ldots, U_{x_J}$ as in Lemma 3, and consider the corresponding homotopies $H_{x_1}, \ldots, H_{x_J}$. Given $u_0 \in W^{s,p}(M; N)$, define inductively $u_j := H_{x_j}(u_{j-1}, 1)$, $j = 1, \ldots, J$, and set

$$H(u_0, t) := \begin{cases} H_{x_1}(u_0, t), & \text{if } 0 \leq t \leq 1, \\ H_{x_2}(u_1, t - 1), & \text{if } 1 < t \leq 2, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ H_{x_J}(u_{J-1}, t - J + 1), & \text{if } J - 1 < t \leq J. \end{cases}$$

Clearly, we have $H \in C^0(W^{s,p}(M; N) \times [0, J]; W^{s,p}(M; N))$ and $H(u_0, 0) = u_0, \forall u_0 \in W^{s,p}(M; N)$. On the other hand, a straightforward induction on j shows that $u_j = 0$ on $U_{x_1} \cup \cdots \cup U_{x_j}$, and thus $H(u_0, J) = u_J = 0$ on $U_{x_1} \cup \cdots \cup U_{x_J} = M$.

To summarize, H is continuous, $H(\cdot, 0) = \mathrm{Id}$ and $H(\cdot, J) = 0$. Thus $W^{s,p}(M; N)$ is contractible.

We continue with the proofs of the auxiliary results used in the proof of Theorem 1.

PROOF OF LEMMA 2. The case $s = 0$ being trivial, we assume that $s > 0$. Since $\|v\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p(\Omega)}$, it remains to prove that

$$|v|_{W^{s,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,p}(\Omega)}. \tag{1}$$

Using a partition of unity, it suffices to consider:

(a) the case where u is supported in a fixed compact $K \subset \Omega$,

(b) the case where u is supported in a fixed small neighborhood of some $x \in \partial\Omega$.

In the first case, we clearly have

$$\begin{aligned} |v|^p_{W^{s,p}(\mathbb{R}^n)} &= |u|^p_{W^{s,p}(\Omega)} + 2\int_K \int_{\mathbb{R}^n\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}}\,dx\,dy \\ &\lesssim |u|^p_{W^{s,p}(\Omega)} + \|u\|^p_{L^p(\Omega)} \sim \|u\|^p_{W^{s,p}(\Omega)}, \end{aligned}$$

and thus (1) holds.

The latter case amounts, after straightening the boundary, to proving Lemma 2 in the case where Ω is replaced by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n;\ x_n > 0\}$, and then the inequality to be proved is

$$I := \int_{\{x_n>0\}} \int_{\{y_n<0\}} \frac{|u(x)|^p}{|x-y|^{n+sp}}\,dx\,dy \lesssim \|u\|^p_{W^{s,p}(\mathbb{R}^n_+)}. \tag{2}$$

By a straightforward scaling argument, we have

$$\int_{\{y_n<0\}} \frac{1}{|x-y|^{n+sp}}\,dy = \frac{C}{x_n^{sp}}, \qquad \forall\, x \in \mathbb{R}^n_+,$$

and thus (2) is equivalent to

$$J := \int_{\mathbb{R}^n_+} \frac{|u(x)|^p}{x_n^{sp}}\,dx \lesssim \|u\|^p_{W^{s,p}(\mathbb{R}^n_+)}. \tag{3}$$

In order to prove (3), which is a cousin of [7, Section 3.2.6, Lemma 1], we proceed to several standard reductions.

(i) The Fubini type inequality

$$\int_{\mathbb{R}^{n-1}} \|u(x', \cdot)\|^p_{W^{s,p}((0,\infty))} \lesssim \|u\|^p_{W^{s,p}(\mathbb{R}^n_+)},$$

see e.g. [1, Lemma 7.44], implies that it suffices to prove (3) when $n = 1$ (and then apply Fubini in the variables $x_1, x_2, \dots, x_{n-1}$).

(ii) The assumption $sp < 1$ implies that $C_c^\infty((0,\infty))$ is dense in $W^{s,p}((0,\infty))$. (This well-known result seems difficult to find in the literature. Let us sketch the argument. We first extend $u \in W^{s,p}((0,\infty))$ to a function $v \in W^{s,p}(\mathbb{R})$ by reflection. We next smooth v and cut-off. This allows us to approximate u with maps $w \in C_c^\infty([0,\infty))$. For such w, let $w_j = \zeta_j\, w$, where $\zeta_j(x) = \zeta(jx)$. Here, $\zeta \in C^\infty([0,\infty))$ is such that $\zeta \equiv 0$ in $[0,1]$ and $\zeta \equiv 1$ in $[2,\infty)$. Then $w_j \in C_c^\infty((0,\infty))$ and, thanks to the assumption $sp < 1$, we have $w_j \to w$ in $W^{s,p}$.) It thus suffices to prove that (3) holds for $u \in C_c^\infty((0,\infty))$. In fact, in this case we have the stronger estimate

$$\int_0^\infty \frac{|u(x)|^p}{x^{sp}}\,dx \lesssim \int_0^\infty \int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}}\,dx\,dy,$$

$\forall\, u \in C_c^\infty((0,\infty))$. When $1 < p < \infty$, this is proved in [7, Section 3.2.6, eq. (6)]. However, the proof in [7] never uses the fact that $p > 1$, and holds also for $p = 1$.

This proves Lemma 2.

For further use, let us note the following straightforward consequences of Lemma 2.

COROLLARY 4. *Let Ω and ω be Lipschitz bounded open sets in $\mathbb{R}^n$ such that $\Omega \subset \omega$. Let $s \geq 0$ and $1 \leq p < \infty$ be such that $sp < 1$. Then the extension operator*

$$W^{s,p}(\Omega;\mathbb{R}) \ni u \mapsto v = \begin{cases} u, & \text{in } \Omega,\\ 0, & \text{in } \omega\setminus\Omega,\end{cases}$$

is continuous from $W^{s,p}(\Omega;\mathbb{R})$ into $W^{s,p}(\omega;\mathbb{R})$.

Consider next, on the Lipschitz manifold M, some point x. Then we may find an open neighborhood U of x and a map $\varphi\colon U \to \widetilde{\omega}$, such that:

(1) $\widetilde{\omega}$ is either the unit ball $B^\ell(0,1)$ in $\mathbb{R}^\ell$, or the upper half of the unit ball in $\mathbb{R}^\ell$ ($\widetilde{\omega} = \{x \in B^\ell(0,1);\ x_\ell \geq 0\}$),

(2) φ is bi-Lipschitz,

(3) $\varphi(x) = 0$.

Define

$$\widetilde{\Omega} := \widetilde{\omega} \cap B^\ell(0,1/2) \qquad \text{and} \qquad U_x := \varphi^{-1}(\widetilde{\Omega}). \tag{4}$$

Note that U_x is an open neighborhood of x.

COROLLARY 5. *Assume that* $0 \in N$. *Let* $s \geq 0$ *and* $1 \leq p < \infty$ *be such that* $sp < 1$. *Then the extension operator*

$$W^{s,p}(U_x; N) \ni u \mapsto T_x u := \begin{cases} u, & \text{in } U_x, \\ 0, & \text{in } M \setminus U_x, \end{cases}$$

is continuous from $W^{s,p}(U_x; N)$ *into* $W^{s,p}(M; N)$.

PROOF. Let $\Omega := \operatorname{int}\big(\widetilde{\Omega}\big)$ and $\omega := \operatorname{int}(\widetilde{\omega})$. Set

$$\overline{T}_x(u) := \begin{cases} u, & \text{in } U_x, \\ 0, & \text{in } U \setminus U_x, \end{cases} \qquad \text{and} \qquad \widetilde{T}_x(v) := \begin{cases} v, & \text{in } \Omega, \\ 0, & \text{in } \omega \setminus \Omega. \end{cases}$$

By Corollary 4, the extension operator $\widetilde{T}_x$ is continuous from $W^{s,p}(\Omega; N)$ into $W^{s,p}(\omega; N)$. Using φ, we find that $\overline{T}_x$ is continuous from $W^{s,p}(U_x; N)$ into $W^{s,p}(U; N)$. On the other hand, it is clear that

$$\|T_x u\|_{W^{s,p}(M)} \leq \|\overline{T}_x u\|_{W^{s,p}(U)} + C\|u\|_{L^p(U_x)},$$

and thus the continuity of $\overline{T}_x$ implies that of T_x, as required.

PROOF OF LEMMA 3. Let U_x be as in (4). Let R_x denote the restriction operator from $W^{s,p}(M)$ into $W^{s,p}(U_x)$ and set $L_x u$ to be u, in $M \setminus U_x$, and 0, in U_x. Then $L_x = \operatorname{Id} - T_x \circ R_x$, and thus L_x is continuous from $W^{s,p}(M)$ into itself.

As in Step 1 in the proof of Theorem 1, consider a homotopy $\widetilde{H} \in C^0(W^{s,p}(\widetilde{\Omega}; N) \times [0, 1]; W^{s,p}(\widetilde{\Omega}; N))$ such that $\widetilde{H}(\cdot, 0) = \operatorname{Id}$ and $\widetilde{H}(\cdot, 1) = 0$. Then $\widetilde{H}$ transfers, via φ, to a homotopy $\overline{H}_x \in C^0(W^{s,p}(U_x; N) \times [0, 1]; W^{s,p}(U_x; N))$ such that $\overline{H}_x(\cdot, 0) = \operatorname{Id}$ and $\overline{H}_x(\cdot, 1) = 0$. Finally, set $H(u, t) := L_x(u) + (T_x \circ \overline{H})(u, t)$. Then H has all the required properties.

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