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BOUNDING SMOOTH SOLUTIONS OF BEZOUT EQUATIONS

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Abstract

Given data $f = (f_1, f_2, \dots, f_n)$ in the holomorphic part $A = F_+$ of a symmetric Banach/topological algebra F on the unit circle $\mathbb{T}$, we estimate solutions $g_j \in A$ to the corresponding Bezout equation $\sum_{j=1}^n g_j f_j = 1$ in terms of the lower spectral parameter δ, $0 < \delta \le |f(z)|$, and an inversion controlling function $c_1(\delta, F)$ for the algebra F. A scheme developed issues from an analysis of the famous Uchiyama-Wolff proof to the Carleson corona theorem and includes examples of algebras of "smooth" functions, as Beurling-Sobolev, Lipschitz, or Wiener-Dirichlet algebras. There is no real "corona problem" in this setting, the issue is in the growth rate of the upper bound for $\|g\|_{A^n}$ as $\delta \to 0$ and in numerical values of the quantities that occur, which are determined as accurately as possible.

1. The effective corona problem in holomorphic algebras

1.1. Holomorphic algebras

Let A be a *holomorphic algebra* on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which means a Banach space of holomorphic functions $A \subset \mathscr{H}ol(\mathbb{D})$ (continuous inclusion) being a *unital topological Banach algebra* in the sense that $1 \in A$ and

$$\|fg\| \le K_1\|f\| \cdot \|g\| \quad (\forall f, g \in A),$$

where $K_1 = K_1(A) > 0$ is a constant. It follows that $A \subset H^\infty(\mathbb{D})$, the algebra of bounded holomorphic functions on $\mathbb{D}$ (and $\|f\|_\infty \le K_1\|f\|_A$ for every $f \in A$). We speak of a *Banach algebra* A, if $K_1(A) = 1$ and $\|1\| = 1$; see the comments in Section 2 below on differences between topological Banach and Banach algebras.

Given an n-tuple $f = (f_j) \in A^n := A \times \cdots \times A$ of functions $f_j \in A$ ("data"), the question of solutions $g = (g_j) \in A^n$ to the corresponding Bezout equation,

$$g \cdot f = \sum_{j=1}^n g_j f_j = 1,$$

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attracted quite a lot of attention because of its interest in complex analysis (interpolation, closed ideals, etc.), as well as an applied importance (say, in H^∞-control theory, in the spectral theory on Sz.-Nagy-Foias model, etc.). For references, see [9], [11], [14], [4].

Originally, Bezout equations were responsible for the so-called "corona problem" (whether the disc $\mathbb{D}$ is dense in maximal ideal space $\mathfrak{M}(A)$) and a classification of finitely generated ideals in A. The techniques of solving these equations, as well as the needs of applications, show that the primary problem is in estimates of solutions g in terms of the data f, even though there is no doubt about the emptyness of the "corona" $\mathfrak{M}(A) \setminus (\operatorname{clos}(\mathbb{D}))$. In order to deal with these estimates, we introduce a norm on A^n as

$$\|f\| = \left(\sum_{j=1}^{n} \|f_j\|_A^2\right)^{1/2} \quad (\text{for } f = (f_j) \in A^n),$$

and the least norm solution for Bezout equations

$$[f]^{-1} := \inf\big\{\|g\| : g \cdot f = 1, g \in A^n\big\}$$

(the inf of an empty set is $+\infty$), and finally, given δ, $0 < \delta \leq 1$, the quantity (possibly infinite)

$$c_n(\delta, A) = \sup\big\{[f]^{-1} : f \in A^n, \|f\| \leq 1, \delta \leq |f(z)| \ (z \in \mathbb{D})\big\},$$

where $|f(z)|^2 = \sum_{j=1}^{n} |f_j(z)|^2$. An "effective corona problem" for an algebra A consists in estimates of $c_n(\delta, A)$ (upper and lower) for $n = 1, 2, \ldots$ and $0 < \delta \leq 1$; $c_1(\delta, A)$ is an effective inversion bound,

$$c_1(\delta, A) = \sup\big\{\|1/f\| : f \in A, \|f\| \leq 1, \delta \leq |f(z)| \ (z \in \mathbb{D})\big\}.$$

As is well-known, the canonical setting for Bezout equations (known as Carleson corona problem/theorem) is related to the Hardy algebra of the disc $H^\infty = H^\infty(\mathbb{D})$, but our approach does not cover this case (since we need a bounded Riesz projection P_+ on a larger symmetric algebra, as L^∞ in this case). The best known estimate for the classical Carleson theorem (elaborated mostly by A. Uchiyama and T. Wolff, 1980) is

$$c_n(\delta, H^\infty) \leq 20 \frac{\log(e\delta^{-1})}{\delta^2},$$

see [11] for references and the proof of this quantitative form.

In this note, for a class of holomorphic algebras A described in Section 1.2 below and called (smoothly) symmetrizable, we give an estimate for $c_n(\delta, A)$

in terms of $c_1(\delta, F)$, where F is a symmetric extension of A. The result is stated in Sections 1.3 and 2.3; Section 2 contains comments on topological Banach algebras. Symmetrizability and other hypotheses of Theorems 1.1 and 2.1 are discussed in Section 3. Some examples (Lipschitz, Beurling-Sobolev, and Wiener-Dirichlet algebras) are considered in Section 4, and the proof of Theorem 1.1 is given in Section 5. Our axiomatic approach is mostly inspired by the paper of O. El-Fallah and M. Zarrabi [8] where the case of Beurling convolution algebras $\ell^1(\mathbb{Z}_+, w_n)$ is settled and the case of Beurling-Sobolev algebras $\ell^p(\mathbb{Z}_+, w_n)$ is suggested (but without entering into the modifications needed for this case; see "Remarque" p. 315 in [8], and a discussion in §§2 and 4.2 below).

1.2. Basic requirements on A

Holomorphic algebras A considered below are "analytic halves" of symmetric function algebras.

A *symmetric function Banach algebra* (respectively, *symmetric topological Banach algebra*) on the unit circle $\mathbb{T}$ is, by definition, a Banach algebra F (respectively, a topological function algebra), $F \subset L^\infty(\mathbb{T})$ (continuous inclusion), containing the trigonometric polynomials $\mathscr{P} := \mathscr{L}\mathrm{in}(z^k : k \in \mathbb{Z})$ and satisfying the following properties:

(A1) $f \in F$ implies $\overline{f} \in F$, $f_* \in F$ and $\|f\|_F = \|\overline{f}\|_F = \|f_*\|_F$, where $f_*(z) = f(\overline{z})$, and $R := \lim_{|n|\to\infty} \|z^n\|_F^{1/n} = 1$;

(A2) the *Riesz projection*

$$P_+\Big(\sum_{j\in\mathbb{Z}} \hat{f}(j)z^j\Big) = \sum_{j\geq 0} \hat{f}(j)z^j \quad (f \in F)$$

($\hat{f}(j)$ stands for a Fourier coefficient) is bounded on F, and Poisson, meaning f_r, $0 \leq r < 1$, $f_r(z) = f(rz)$, define contractive maps on $A = F_+ = P_+(F)$,

$$\|f_r\|_A \leq \|f\|_A.$$

Clearly, the analytic half of a symmetric algebra,

$$A = F_+ = P_+(F),$$

is (after a natural extension of $P_+ f$ to the disc $\mathbb{D}$) a unital holomorphic topological Banach algebra. The next requirement (A3) states a kind of sequential weak completeness of A, as follows.

(A3) If $f_k \in A = F_+$, $\sup_{k\geq 1} \|f_k\|_A < \infty$ and the limit $f(z) = \lim_k f_k(z)$ $(\forall z \in \mathbb{D})$ exists, then $f \in A$.

From now on, we fix the notation

$$F \quad \text{and} \quad A = F_+$$

for algebras satisfying (A1)–(A3). Later, we give several examples of such algebras F and A, but for now we only mention that the classical Wiener algebra $F = W$,

$$W = \left\{ f = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n : \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty \right\},$$

satisfies (A1)–(A3), but $F = L^\infty(\mathbb{T})$ does not (P_+ is not bounded). We notice a few immediate consequences of the definitions.

(1) The *limit* R in (A1) exists and is the spectral radius of the function z;

(2) The *Szegő kernels* $k_\lambda = 1/(1 - \lambda z)$ are in A for every λ, $\lambda \in \mathbb{D}$. Also, every L^∞ function which is real analytic on $\mathbb{T}$ is in F (in particular, the Poisson means f_r of a function $f \in F$ are automatically in F). The contractivity of $f \mapsto f_r$ is just a technical detail, which permits to avoid some unpleasant computations of constants; it is satisfied for any rotation invariant algebra F: $f \in F \Rightarrow f_\zeta \in F$ and $\|f_\zeta\|_F = \|f\|_F$, where $f_\zeta(z) = f(\overline{\zeta} z)$, $\forall \zeta \in \mathbb{T}$.

(3) The *backward shift* operator

$$S^* f = P_+(\overline{z} f) = \frac{f - f(0)}{z}$$

is bounded on A. Moreover, given a holomorphic function $h \in \mathcal{H}\mathrm{ol}(\mathbb{D})$, $h = \sum_{k \geq 0} \hat{h}(k) z^k$, and an analytic polynomial $f \in \mathcal{P}_a := \mathcal{L}\mathrm{in}(z^k : k \in \mathbb{Z}_+)$, the function $h(S^*) f = \sum_{k \geq 0} \hat{h}(k) (S^*)^k f$ is well defined and is a polynomial, so that $h(S^*)\colon \mathcal{P}_a \to \mathcal{P}_a$ is a linear mapping. We define the *functional calculus algebra* of S^* as the set

$$M(A) = \{h \in \mathcal{H}\mathrm{ol}(\mathbb{D}) : h(S^*)\colon \mathcal{P}_a \to \mathcal{P}_a \text{ extends to a bounded map } A \to A\}.$$

Clearly, $M(A)$ equipped with the operator norm

$$\|h\|_M = \|h(S^*)\|_{A \to A}$$

is a unital Banach algebra contractively embedded into H^∞ (since $h(S^*) k_\lambda = h(\lambda) k_\lambda$ for every $\lambda \in \mathbb{D}$). In fact, $h(S^*)$ is nothing but an anti-analytic Toeplitz operator on A.

1.3. Main theorem

THEOREM 1.1. *Let F and G be* (A1)–(A3) *Banach algebras on the unit circle $\mathbb{T}$ satisfying*

$$F \subset G \quad \textit{and} \quad G_+ \subset M(F_+)$$

(*continuous embeddings*), *and the* (*embedding type*) *condition:*

$$\text{(A4)} \qquad \int_0^1 \|\varphi_r'\|_{G_+}\, dr \le K_4 \|\varphi\|_{F_+} \quad \textit{for every } \varphi \in F_+ = A,$$

where $K_4 = K_4(F_+, G_+) > 0$ is a constant. Then, the following estimate holds for Bezout equations in $A = F_+$:

$$c_n(\delta, A) \le K c_1(\delta^2, F)^2$$

for all $0 < \delta \le 1$ and all $n = 1, 2, \ldots$, where K is a constant depending on F and G. In particular,

$$K \le \|P_+\|_F + 4\|S^*\|_{A \to A} K_2^3 K_3 K_4 \|P_+\|_G := \overline{K}$$

where K_2 is an embedding constant $\|f\|_G \le K_2 \|f\|_F$ $(\forall f \in F)$, K_3 a functional calculus constant $\|h(S^*)\| = \|h\|_{M(F_+)} \le K_3 \|h\|_{G_+}$ $(\forall h \in G_+)$, and $\|P_+\|_F = \|P_+\|_{F \to F}$, $\|P_+\|_G = \|P_+\|_{G \to G}$.

In the case when all involved mappings are contractions (i.e., $\|S^*\|_{A \to A} = \|P_+\|_F = K_2 = K_3 = K_4 = \|P_+\|_G = 1$), we have $\overline{K} = 5$; see Section 4 for examples.

Condition (A4) of Theorem 1.1 looks rather technical but it should be considered as a requirement on a holomorphic algebra A allowing estimatation of solutions of Bezout equations in A. The theorem can also be stated as follows.

THEOREM 1.2. *If A is the holomorphic part $A = F_+$ of a symmetric algebra F, stable under the backward shift S^* ($S^* A \subset A$), and the functional calculus for S^* is so good that there exists a symmetric algebra G such that $G_+ \subset M(A)$ and*

$$\int_0^1 \|\varphi_r'\|_{G_+}\, dr \le K_4 \|\varphi\|_A \quad (\forall \varphi \in A),$$

where $K_4 > 0$ and φ_r stands for the Poisson mean of φ, then solutions of Bezout equations in A allow an estimate $c_n(\delta, A) \le K c_1(\delta^2, F)^2$ $(0 < \delta \le 1;$ $n = 1, 2, \ldots)$.

Below, we largely comment on property (A4) of Theorem 1.1. In particular, we analyze possibilities of the "extreme" choices $G_+ = A$ and $G_+ = M(A)$, and show that (A4) is an improved form of the inclusion $A \subset M(A)$, which, in turn, is equivalent to a "symmetrizability" of A (see Section 3). In the next Section 2, we also show how to modify the claim of Theorem 1.1 when replacing "Banach algebras" by "topological Banach algebras". The principal examples (presented in Section 4) deal just with topological Banach algebras rather than Banach algebras.

2. Remarks on topological Banach algebras

2.1. The scaling of the constant $c_n(\delta, X)$ for equivalent norms

Suppose we have two equivalent norms on an algebra X, say,

$$a\|f\| \le \|f\|_* \le b\|f\| \quad (\forall f \in X),$$

for some $a > 0, b > 0$. Taking f from the definition of $c_n(\delta, X)$, i.e. $\|f\| \le 1$, $|f| \ge \delta > 0$, we get $\|f/b\|_* \le 1$, $|f/b| \ge \delta/b > 0$, and hence $\|f^{-1}\| = (1/b)\|(f/b)^{-1}\| \le (1/ab)\|(f/b)^{-1}\|_*$, i.e.

$$c_n(\delta, X) \le \frac{1}{ab} c_n(\delta/b, X_*),$$

and symmetrically,

$$\frac{1}{ab} c_n(\delta/a, X_*) \le c_n(\delta, X).$$

We apply this remark to topological Banach algebras.

2.2. Topological Banach algebras versus Banach algebras

A *unital topological Banach algebra X* is a Banach space and a unital algebra with a continuous multiplication, i.e. $1 \in X$ and

$$\|fg\| \le K_1\|f\| \cdot \|g\| \quad (\forall f, g \in X),$$

where $K_1 = K_1(X) > 0$ is a constant. There exists an equivalent norm $\|\cdot\|_*$ which makes $X_* = (X, \|\cdot\|_*)$ a Banach algebra, which means that $K_1(X_*) = 1$. For example,

$$\|f\|_* = \|M_f\|,$$

where $M_f g = fg$ is a multiplication operator $X \to X$. In this case,

$$\frac{1}{\|1\|}\|f\| \le \|f\|_* \le K_1\|f\| \quad (\forall f \in X),$$

and hence, from §2.1,

$$\frac{\|1\|}{K_1} c_n(\delta\|1\|, X_*) \le c_n(\delta, X) \le \frac{\|1\|}{K_1} c_n(\delta/K_1, X_*).$$

The following statement shows how the estimate of Theorem 1.1 can vary when dealing with topological Banach algebras instead of Banach algebras. It will be useful, in particular, for Beurling-Sobolev topological Banach algebras $\mathscr{F}\ell^p(\mathbb{Z}, w_n)$ (see §4.2) which are never Banach algebras for $p > 1$. In the case when the inverse controlling function $\delta \mapsto c_1(\delta)$ grows rapidly as $\delta \downarrow 0$, the most important difference between Theorem 1.1 and Theorem 2.1 below is, of course, in the argument scaling factor $K_1(F)^2\|1\|_F$.

2.3. *Theorem 1.1 for topological Banach algebras*

THEOREM 2.1. *With the notation and hypotheses of Theorem 1.1, assume that F and G are topological Banach algebras on the unit circle $\mathbb{T}$. Then,*

$$c_n(\delta, A) \leq \tilde{K} c_1\left(\frac{\delta^2}{K_1(F)^2\|1\|_F}, F\right)^2$$

for all $0 < \delta \leq 1$ *and all* $n = 1, 2, \ldots$, *where*

$$\begin{aligned}\tilde{K} &= \tilde{K}(F, G) := \frac{K_1(F)}{\|1\|_F}\overline{K}_*,\\ \overline{K}_* &= K_1(F)\|P_+\|_F\|1\|_F\\ &\qquad + 4K_1(F)K_1(G)^5K_2^3K_3K_4\|1\|_F^5\|1\|_G^2\|P_+\|_G\|S^*\|_{A_*},\end{aligned}$$

$K_2 = K_2(F \to G)$, $K_3 = K_3(G_+ \to M(F_+))$ *and* $K_4 = K_4(G, F)$.

PROOF. It easily follows from the remarks of §2.2 above and Theorem 1.1 applied to Banach algebras E_* and F_*; we only need to recalculate the embedding constants

$$K_2 = K_2(F \to G) \quad \text{and} \quad K_3 = K_3(G_+ \to M(F_+)).$$

Omitting the details, we get $K_{2*} := K_2(F_* \to G_*) \leq K_1(G)K_2\|1\|_F$, $K_{3*} := K_3(G_{*+} \to M(F_{*+})) \leq K_3K_1(F)\|1\|_G\|1\|_F$, and then by Theorem 1.1 $c_n(\delta, F_{*+}) \leq K_*c_1(\delta^2, F_*)^2$, where

$$\begin{aligned}K_* &\leq \|P_+\|_{F_*} + 4K_{2*}^3K_{3*}K_{4*}\|P_+\|_{G_*}\|S^*\|_{A_*}\\ &\leq K_1(F)\|P_+\|_F\|1\|_F\\ &\qquad + 4K_1(F)K_1(G)^5K_2^3K_3K_4\|1\|_F^5\|1\|_G^2\|P_+\|_G\|S^*\|_{A_*} =: \overline{K}_*.\end{aligned}$$

Finally, using §2.2, we get the estimate stated in Theorem 2.1.

REMARK 2.2. Of course, in the case of a Banach algebra F (i.e., $\|1\|_F = K_1(F) = 1$), we have $K_* = K$ and $\overline{K}_* = \overline{K}$. One more observation is that we always have

$$\|S^*\|_{A_*} \leq 2\|S^*\|_A;$$

indeed, taking $f, g \in A$ we obtain

$$gS^*f = g\frac{f - f(0)}{z} = \frac{gf - (gf)(0)}{z} - f(0)\frac{g - g(0)}{z}$$
$$= S^*(fg) - f(0)S^*g,$$

and hence $\|gS^*f\|_A \leq \|S^*(fg)\|_A + |f(0)| \cdot \|S^*g\|_A \leq \|S^*\|_A\|f\|_*\|g\|_A + |f(0)| \cdot \|S^*\|_A\|g\|_A \leq 2\|S^*\|_A\|f\|_*\|g\|_A$, which means $\|S^*f\|_* \leq 2\|S^*\|_A\|f\|_*$, and the claim follows.

Yet another case occurring in applications (see §4.2 below) is the "contractive case" of a topological Banach algebra A, where $\|S^*\|_{A\to A} = \|P_+\|_F = \|P_+\|_G = \|1\|_G = \|1\|_F = K_1(G) = K_2 = K_3 = K_4 = 1$, and hence $\tilde{K} \leq 9K_1(F)^2$.

3. Symmetrizable algebras and embedding condition (A4)

3.1. Symmetrizable holomorphic algebras

Commenting on condition (A4) of the theorem, notice that there φ'_r means $(\varphi')_r = \sum_{k\geq 1}\hat{\varphi}(k)kr^{k-1}z^{k-1}$, so that

$$S^*\varphi(z) = \frac{\varphi(z) - \varphi(0)}{z} = \sum_{k\geq 1}\hat{\varphi}(k)z^{k-1} = \int_0^1 \varphi'_r(z)\,dr,$$

$$\varphi(z) = \varphi(0) + z\int_0^1 \varphi'_r(z)\,dr \quad (\forall z \in \mathbb{D}),$$

and then

$$\|\varphi\|_M \leq |\varphi(0)| + \|z\|_M \int_0^1 \|\varphi'_r\|_M\,dr \leq \|\varphi\|_A + \|S^*\|K_2K_3\|\varphi\|_A.$$

Consequently,

$$(\mathrm{A4}) \;\;\Rightarrow\;\; A \subset M(A),$$

and

$$\|\varphi\|_M \leq (1 + K_2K_3\|S^*\|)\|\varphi\|_A \quad (\forall\varphi \in A).$$

The claim follows.

We say that a holomorphic algebra A is *symmetrizable* if there exists a symmetric topological Banach algebra B such that

$$A = P_+(B) =: B_+$$

(with equivalent norms); B is called a *symmetric extension* of A.

Clearly, for a symmetrizable algebra A, there exists a minimal symmetric extension $B = SA$, where

$$SA = A + \overline{A}_0$$

(the direct sum), $\overline{A}_0$ stands for complex conjugate, $A_0 = \{f \in A : \hat{f}(0) = 0\}$, and the norm is defined as a symmetrization of the norm $\|g+\overline{h}\| = \|g\|_A + \|h\|_A$ ($g \in A$, $h \in A_0$),

$$\|f\|_B = \tfrac{1}{4}(\|f\| + \|\overline{f}\| + \|f_*\| + \|\overline{f}_*\|).$$

In this case, $\|P_+ \colon SA \to SA\| = 1$.

3.2. *A theorem*

THEOREM 3.1. *A holomorphic algebra A is symmetrizable if and only if $A \subset M(A)$.*

PROOF. If there exists a symmetric extension B of A, then

$$\|h(S^*)f\|_A = \|P_+(h_*f)\|_A \le \alpha \|P_+\|_{B\to B} \cdot \|h_*f\|_B \le \beta \|h\|_A \|f\|_A$$

for all $h, f \in A$ (and some constants $\alpha, \beta > 0$), and hence $h \in M(A)$.

Conversely, if $A \subset M(A)$, we can set, as before, $B = SA = A + \overline{A}_0$ endowed with the above mentioned symmetrization $\|\cdot\|_B$ of the norm $\|g + \overline{h}\| = \|g\|_A + \|h\|_A$ for $g \in A$, $h \in A_0$. Then $A = B_+$ (with an equivalence of norms, because the norms $f \mapsto \|f\|_A$ and $f \mapsto \|\overline{f}_*\|_A$ are equivalent on A) and B is a topological Banach algebra: if $g + \overline{h}, k + \overline{\ell} \in B = A + \overline{A}_0$, then

$$(g + \overline{h})(k + \overline{\ell}) = gk + P_+(g\overline{\ell} + \overline{h}k) + \overline{h\ell} + P_-(g\overline{\ell} + \overline{h}k),$$

where $P_- = I - P_+$, so that $\|P_+(g\overline{\ell})\| = \|\overline{\ell}_*(S^*)g\| \le c\|\ell\|_A\|g\|_A$, and similarly for other mixed products; thus $\|(g+\overline{h})(k+\overline{\ell})\| \le c(\|g\|+\|h\|)(\|k\|+\|\ell\|) \le c\|g + \overline{h}\| \cdot \|k + \overline{\ell}\|$, where $c > 0$ are constants (maybe different), and the claim follows.

3.3. *Symmetrization and condition* (A4)

A symmetrizable holomorphic algebra A has a symmetric extension $F \supset A = F_+$ and satisfies a functional calculus estimate $\|\varphi\|_{M(A)} \le C\|\varphi\|_A$. If the latter can be improved up to inequalities (A4),

$$\|\varphi\|_{M(A)} \le c\int_0^1 \|\varphi_r'\|_{G_+}\,dr \le d\|\varphi\|_{F_+} \quad (\forall \varphi \in F_+)$$

with a weaker symmetric norm $\|\cdot\|_G$, then Theorem 1.1 may be applied and we get an estimate for $c_n(\delta, A)$. Therefore, looking for a candidate for an algebra G we consider *symmetrizable* intermediate holomorphic algebras G_+,

$$F_+ \subset G_+ \subset M(F_+).$$

We start with a list of examples (anticipating exact definitions and justifications, for which we refer to next Section 4).

3.4. Examples of working with condition (A4)

3.4.1. Beurling-Sobolev algebras $A := \mathscr{F}\ell^p(\mathbb{Z}_+, w_n) = (\mathscr{F}\ell^p(\mathbb{Z}, w_n))_+$. (See §4.2 for the definition) are symmetrizable under a (very weak) condition that the weight function is monotone ($w_{n+1} \geq w_n$), or quasi-monotone ($\sup_{0\leq k\leq n} \frac{w_k}{w_n} < \infty$). A natural choice of G in order to satisfy (A4) is $G = W = \mathscr{F}\ell^1(\mathbb{Z})$, see §4.2.

3.4.2. H^∞ is not symmetrizable. Indeed, the norms $\|S^{*n}\|_{H^\infty\to H^\infty} \approx \log n$ cannot be bounded by $\|z^n\|_{H^\infty} = 1$. In fact, $M(H^\infty)$ is the algebra of Cauchy-Stieltjes multipliers $\{\varphi \in \mathscr{H}\mathrm{ol}(\mathbb{D}) : \varphi \cdot (L^1/H^1_-) \subset L^1/H^1_-\}$, and in particular, $\|\varphi(S^*)\|_{H^\infty\to H^\infty} \leq c\|\varphi\|_{H^1_1}$, where $H^1_1 = \{\varphi \in \mathscr{H}\mathrm{ol}(\mathbb{D}) : \varphi' \in H^1\}$, see [18].

3.4.3. Lipschitz holomorphic algebra $A = \mathrm{Lip}(\mathbb{T}, \alpha)_+$, $0 < \alpha < 1$. (See §4.3 for the definition), is symmetrizable (since $\mathrm{Lip}(\mathbb{T}, \alpha)$ is symmetric). Here $M(A) = H^\infty$ (see point §3.4.5 below), which is not symmetrizable, but one can take an intermediate $G = \mathrm{Lip}(\mathbb{T}, \varepsilon)$ with a sufficiently small $\varepsilon > 0$ in order to get condition (A4), see §4.3 below for details.

3.4.4. About the choice $G = F$. Taking $G = F$, where F is a symmetric topological Banach algebra, we get $G_+ = F_+ \subset M(F_+)$, but an embedding inequality (A4)

$$\int_0^1 \|\varphi_r'\|_{F_+}\, dr \leq K_4\|\varphi\|_{F_+} \quad (\forall \varphi \in F_+)$$

seems to hold very seldom. As we will see (Section 4), it is true for Wiener-Beurling algebras $F = \mathscr{F}\ell^1(\mathbb{Z}, w_n)$, but it fails for other interesting algebras, as in the following examples.

(1) (A4) fails with $G = F$ for Beurling-Sobolev algebras $F = \mathscr{F}\ell^p(\mathbb{Z}, w_n)$ with $p > 1$: indeed, let us compare the left/right hand sides of the inequality for $\varphi_\varepsilon = \sum_{k\geq 1} \frac{1}{k^{(1+\varepsilon)/p} w_k} z^k$, $\varepsilon > 0$ as $\varepsilon \to 0$:

$$\|\varphi_\varepsilon\| = \left(\sum_{k\geq 1} \frac{1}{k^{1+\varepsilon}}\right)^{1/p} \approx (1/\varepsilon)^{1/p} \quad \text{as } \varepsilon \to 0,$$

whereas (with constants $c > 0$ which may vary from one expression to another)

$$\int_0^1 \|\varphi_r'\|\,dr = \int_0^1 \Big(\sum_{k\geq 1} \frac{k^p r^{p(k-1)}}{k^{1+\varepsilon}} \Big(\frac{w_{k-1}}{w_k}\Big)^p\Big)^{1/p}$$

$$\geq c\int_0^1 \Big(\sum_{k\geq 1} \frac{k^p r^{p(k-1)}}{k^{1+\varepsilon}}\Big)^{1/p} dr \geq c\int_0^1 \Big(\frac{1}{(1-r^p)^{p-\varepsilon}}\Big)^{1/p} dr$$

$$\geq c\int_0^1 \frac{1}{(1-r)^{1-(\varepsilon/p)}}\,dr \geq \frac{c}{\varepsilon} \quad \text{as } \varepsilon \to 0,$$

and the claim follows.

(2) (A4) fails with $G = F$ for the Lipschitz algebras $F = \mathrm{Lip}(\mathbb{T}, \alpha)$, $0 < \alpha < 1$: indeed, a classical result of Hardy-Littlewood (1932) tells us that the norms

$$\|f\|_\alpha = \|f\|_{L^\infty(\mathbb{D})} + \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha},$$

and

$$\|f\| = \|f\|_{L^\infty(\mathbb{D})} + \sup_{|z|<1} |f'(z)|(1-|z|)^{1-\alpha}$$

are equivalent for holomorphic functions for $0 < \alpha < 1$; see for example §5.2 of [5]. Now, take $\varphi(z) = \int_0^z (1-\zeta)^{\alpha-1}\,d\zeta$; then $\|\varphi\| < \infty$ but

$$\int_0^1 \|\varphi_r'\|\,dr \geq \int_0^1 |\varphi''(r^2)|(1-r)^{1-\alpha}\,dr \geq (1-\alpha)\int_0^1 (1-r)^{-1}\,dr = \infty,$$

and the claim follows.

Notice that both $\mathscr{F}\ell^p(\mathbb{Z}, w_n)$ and $\mathrm{Lip}(\mathbb{T}, \alpha)$ *satisfy* (A4) condition, but with different choices of the algebra G (see Section 4).

3.4.5. About the choice $G_+ = M(F_+)$. Given a symmetric algebra F, the inequality

$$\int_0^1 \|\varphi_r'\|_{M(F_+)}\,dr \leq K_4\|\varphi\|_{F_+} \quad (\forall\varphi \in F_+)$$

is the weakest embedding requirement in the series of (A4) conditions. It may happen that it is always true (we have no *symmetric algebra* counterexamples; see point (3) below for a non-symmetric one) but it would be useful for Theorem 1.1/2.1 only if the algebra $M(F_+)$ is symmetrizable. Let us give a couple of examples.

(1) For a Beurling-Sobolev algebra $F = \mathscr{F}\ell^p(\mathbb{Z}, w_n)$ with a "quasi-monotone" weight (w_n) (see §4.2 below) satisfying (A1), embedding (A4) holds

with $G_+ = M(F_+)$: indeed, F is backward stable (see §4.1) and hence $F_+ \subset W_+ \subset M(F_+)$ and (A4) follows (see §§4.1–4.2 below).

(2) For the Lipschitz algebra $F = \mathrm{Lip}(\mathbb{T}, \alpha)$, $0 < \alpha < 1$, we have

$$M(F_+) = M(\mathrm{Lip}(\mathbb{T}, \alpha)_+) = H^\infty,$$

which is not symmetrizable (see §3.4.2 above), but however (A4) holds for $G_+ = M(F_+)$: indeed, it is known that F_+ is a dual space to a quasi-normed Hardy space $H^p(\mathbb{D})$, $p = \frac{1}{1+\alpha}$, in the sense that the norm $\|f\|_{\mathrm{Lip}(\alpha)_+}$ is equivalent to

$$\|f\| = \sup\left\{\left|\sum_{k\geq 0} \hat{g}(k)\hat{f}(k)\right| : \|g\|_p = 1\right\},$$

where $g \in \mathscr{P}_a$ and $\|g\|_p = \sup_{0<r<1}\left(\int_{\mathbb{T}} |g(r\zeta)|^p \, dm(\zeta)\right)^{1/p}$ (B. W. Romberg, 1960; see Theorem 7.5 in [5]). Since the backward shift $S^*\colon F_+ \to F_+$ is the adjoint operator to the shift $S\colon H^p(\mathbb{D}) \to H^p(\mathbb{D})$ $(Sg = zg)$, we obtain that $\varphi(S^*)$ is bounded on $\mathrm{Lip}(\mathbb{T}, \alpha)_+$ if and only if the multiplication operator $M_\varphi g = \varphi g$ is bounded on $H^p(\mathbb{D})$, that is, if and only if $\varphi \in H^\infty$. Therefore, $M(\mathrm{Lip}(\mathbb{T}, \alpha)_+) = H^\infty$.

In order to check (A4) with $G_+ = M(\mathrm{Lip}(\mathbb{T}, \alpha)_+) = H^\infty$, we use the same Hardy-Littlewood theorem as in §3.4.4(2) above: if $\varphi \in F_+ = \mathrm{Lip}(\mathbb{T}, \alpha)_+$, then $|\varphi'(z)| \leq c\|\varphi\|_{\mathrm{Lip}(\alpha)_+}(1-|z|)^{\alpha-1}$, so that

$$\int_0^1 \|\varphi_r'\|_{M(F_+)} \, dr \leq c\|\varphi\|_{\mathrm{Lip}(\alpha)_+} \int_0^1 (1-r)^{\alpha-1} \, dr = C\|\varphi\|_{\mathrm{Lip}(\alpha)_+},$$

where $c > 0$, $C > 0$ are constants, and the claim follows.

Later, we will show that for $F = \mathrm{Lip}(\mathbb{T}, \alpha)$, $0 < \alpha < 1$, there exists a symmetric G satisfying (A4).

(3) For a non-symmetric algebra $F = L^\infty(\mathbb{T})$, (A4) fails (even with $G_+ = M(F_+)$): indeed, it is well-known that there exists $\varphi \in F_+ = H^\infty$ such that $\int_0^1 |\varphi'(r)| \, dr = \infty$ (i.e., φ is a mapping for which the curve $\varphi([0, 1))$ has infinite length), and $\|\varphi_r'\|_{M(F_+)} \geq \|\varphi_r'\|_\infty \geq |\varphi'(r)|$, so that $\int_0^1 \|\varphi_r'\|_{M(F_+)} \, dr = \infty$, and the claim follows.

See also §3.4.2 on the algebra $M(H^\infty)$.

4. Backward stable, Beurling-Sobolev and Lipschitz algebras, and other examples

Here we give some examples of algebras satisfying (A1)–(A4), and so the conclusion of Theorem 1.1/2.1. We start with a general remark on a class of algebras satisfying (A1)–(A4).

4.1. Backward stable algebras satisfying (A1)–(A3)

A holomorphic algebra A is *backward stable* if

$$K_5 := \sup_{n\geq 0} \|S^{*n}\|_{A\to A} < \infty.$$

LEMMA 4.1. *Let A be a backward stable algebra satisfying* (A1)–(A3) *and embedded into the Wiener holomorphic algebra, $A \subset W_+$, with*

$$\|f\|_{W_+} \leq K_6 \|f\|_A \quad (\forall f \in A).$$

Then A satisfies (A4) *with $G_+ = W(\mathbb{T})_+$* (*and with $G_+ = M(A)$*) *and*

$$K_4(A, M(A)) \leq K_5 K_6, \quad K_4(A, W_+) \leq K_6.$$

PROOF. Indeed, it is clear that $\|h\|_M = \|h(S^*)\| = \left\|\sum_{k\geq 0} \hat{h}(k) S^{*k}\right\| \leq K_5 \|h\|_W$ ($\forall h \in W_+$), so that $W_+ \subset M(A)$, and moreover

$$\int_0^1 \|h_r'\|_M \, dr \leq K_5 \int_0^1 \|h_r'\|_W \, dr = K_5 \int_0^1 \sum_{k\geq 0} |\hat{h}(k)| k r^{k-1} \, dr \leq$$
$$\leq K_5 \|h\|_W \leq K_5 K_6 \|h\|_A,$$

and the claim follows.

REMARK 4.2. If, in the above computations, $G = W$ and F is a *symmetric algebra* and $A = F_+$ then in the notation of Theorem 1.1/2.1, we have

$$K_6 \leq K_2(F \to G), \quad K_5 = K_3(G_+ \to M(A)).$$

4.2. Example: Beurling-Sobolev algebras satisfy (A1)–(A4)

Given positive numbers $w_k > 0$, we denote by $\ell^p(\mathbb{Z}, w_n)$ a weighted space

$$\ell^p(\mathbb{Z}, w_n) = \{x = (x_j)_{j\in\mathbb{Z}} : (x_j w_j) \in \ell^p(\mathbb{Z})\}, \quad 1 \leq p \leq \infty,$$

equipped with the norm $\|x\| = (\sum_{k\in\mathbb{Z}} |x_k w_k|^p)^{1/p}$ (with the usual modification for $p = \infty$). If

$$w_k = w_{-k} \quad (k \in \mathbb{Z}),$$

the Banach space $\ell^p(\mathbb{Z}, w_n)$ is symmetric with respect to involutions $x \mapsto \overline{x}$ and $x \mapsto x_* = (x_{-j})_{j\in\mathbb{Z}}$. For each finitely supported sequence x, its (discrete) Fourier transform is

$$\mathscr{F}(x_j) = f(z) = \sum_{j\in\mathbb{Z}} x_j z^j, \quad |z| = 1.$$

Assuming

$$R = \lim_{k\to\infty} w_k^{1/k} = 1,$$

one can see that $\ell^p(\mathbb{Z}, w_n)$ is a convolution topological Banach algebra on $\mathbb{Z}$ if and only if the mapping $\mathscr{F}$ extends to a continuous embedding $\mathscr{F}\colon \ell^p(\mathbb{Z}, w_n) \to C(\mathbb{T})$ and

$$F := \mathscr{F}\ell^p(\mathbb{Z}, w_n)$$

is a (topological) function algebra on the circle $\mathbb{T}$ (i.e., $\|fg\| \le K_1\|f\| \cdot \|g\|$ for every $f, g \in \mathscr{F}\ell^p(\mathbb{Z}, w_n)$, where $\|\mathscr{F}f\| = \|f\|_{\ell^p(w_j)}$); it is called a *Beurling-Sobolev algebra* (BS algebra, for short). In fact, it is easy to see that if $\mathscr{F}\ell^p(\mathbb{Z}, w_n) \subset C(\mathbb{T})$, then already

$$\mathscr{F}\ell^p(\mathbb{Z}, w_n) \subset W(\mathbb{T})$$

with the same embedding constant

$$K_1 = K_2(F, W(\mathbb{T})).$$

For these and other basic properties of BS algebras, as well as for estimates for $c_1(\delta, A)$, see [7], [13]. Similar definitions and properties hold for *holomorphic BS algebras*

$$A := \mathscr{F}\ell^p(\mathbb{Z}_+, w_n) = (\mathscr{F}\ell^p(\mathbb{Z}, w_n))_+.$$

Recall that for $p = 1$, a necessary and sufficient condition for $\mathscr{F}\ell^p(\mathbb{Z}_+, w_n)$ to be a BS algebra is the (classical) submultiplicativity property

$$w_{j+k} \le K_1 w_j w_k$$

(it is simply a Banach algebra, if $w_0 = 1$ and $w_{j+k} \le w_j w_k$). For $1 < p \le \infty$, no reasonable necessary and sufficient condition is known, but for "regular" weights (w_j) (for example, if $w_j \ge 1$ and $(\log w_j)$ is concave or convex, see details in [7]) the condition in question does exist:

$$(1/w_j) \in \ell^{p'}(\mathbb{Z}_+), \qquad \frac{1}{p'} + \frac{1}{p} = 1,$$

or equivalently, $\ell^p(\mathbb{Z}_+, w_n) \subset \ell^1(\mathbb{Z}_+)$, or $\mathscr{F}\ell^p(\mathbb{Z}_+, w_n) \subset W_+$.

Speaking of numerical values of the above constants $K_j = K_j(A)$ for BS algebras (which participate in estimates of $c_n(\delta, A)$), recall that (see [10])

$$K_1(A) \le \sup_{n\ge 0}\left(\sum_{k=0}^{n}\left(\frac{w_n}{w_k w_{n-k}}\right)^{p'}\right)^{1/p'},$$

where $\frac{1}{p}+\frac{1}{p'}=1$ (this also gives yet another sufficient condition for $\ell^p(\mathbb{Z}_+,w_n)$ to be a topological convolution Banach algebra, see [10]). The sharpness of this estimate is known (and obvious) for $p=1$ only; for $p>1$, whatever is the weight (w_n), we always have $K_1(A)>1$ (see [7]). So, in this case, a renormalization of §2.2 is necessary. Constants K_5 and K_6 can be (obviously) exactly computed as

$$K_5(A)=\sup_{n,k\geq 0}\frac{w_n}{w_{n+k}},\qquad K_6(A)=\|(1/w_n)_{n\geq 0}\|_{\ell^{p'}(\mathbb{Z}_+)},$$

so that $K_4(A,W_+)\leq K_5$ (by §3.1 above). $K_5=1$ means that $(w_n)_{n\geq 0}$ is a non decreasing sequence; a weight with $K_5(A)<\infty$ is said to be *quasi-monotone*.

Consequently, Theorem 1.1/2.1 applies to a BS algebra $A=\mathscr{F}\ell^p(\mathbb{Z}_+,w_n)$, if

$$K_5(A)<\infty\quad\text{and}\quad K_6(A)<\infty$$

(as is mentioned above, the latter condition implies already $K_1(A)<\infty$ if (w_n) is "regular"), with auxiliary algebras

$$F=\mathscr{F}\ell^p(\mathbb{Z},w_n)\quad\text{and}\quad G=W(\mathbb{T})=\mathscr{F}\ell^1(\mathbb{Z}).$$

Since $G=W(\mathbb{T})$, we have $K_3(A,W_+)=K_5(A)$ and

$$K_4(F_+,W_+)=K_6(A)\leq K_2(F)=\|(1/w_n)_{n\in\mathbb{Z}}\|_{\ell^{p'}(\mathbb{Z})}.$$

If (w_n) is a non-decreasing submultiplicative sequence, $w_0=1$ and $p=1$, then all $K_j=1$.

Under the "regularity" conditions on (w_n) mentioned above, the norm inversion constant $c_1(\delta,F)$ is known to be finite for all $0<\delta\leq 1$ whenever $F=\mathscr{F}\ell^p(\mathbb{Z},w_n)\neq\mathscr{F}\ell^1(\mathbb{Z})=W$ (so, if $p>1$, or $p=1$ and $\lim_n w_n=\infty$). Numerical estimates for $c_1(\delta,F)$ (strongly depending on (w_n)) are given in [7], [6], [16], [2]. Applying Theorem 1.1/2.1, we use $G=W(\mathbb{T})$, $\|P_+\|_F=\|P_+\|_G=K_1(G)=1$, $\|1\|_F=w_0$, as well as the above relations between the constants K_j. *Assuming for simplicity that* $w_0=1$ *and* (w_n) *is monotone* $(K_3=K_5=1)$, we obtain

$$c_n(\delta,\mathscr{F}\ell^p(\mathbb{Z},w_n^\alpha))\leq\tilde{K}c_1(\delta^2/K_1(F)^2,F)^2,$$

where K_1 and $\tilde{K}$ are bounded as above in terms of the constants K_j; in particular, following the case "Banach algebra" or "topological Banach algera", we have

$$\tilde{K}\leq K_1(F)^2\big(1+4K_2(F)^4\big)\quad\text{or}\quad\tilde{K}\leq K_1(F)^2\big(1+8K_2(F)^4\big).$$

It follows from previous comments (see also [7], [13]), that the constants $K_1(F)$ and $K_2(F)$, *for "regular" weights* (w_n), have the same order of magnitude, in which case we have

$$\tilde{K} \le a K_2(F)^6,$$

where $a > 0$ stands for an absolute constant.

Numerical examples. (1) The case $p = 1$. If $(w_n)_{n\ge 0}$ is a non-decreasing submultiplicative sequence, $w_0 = 1$ and $p = 1$, then all $K_j = 1$, and we have

$$c_n(\delta, A) \le 5c_1(\delta^2, F)^2, \quad 0 < \delta \le 1$$

(a result obtained earlier in [8]).

For $p = 1$ and bounded (w_n), we have $A = \mathscr{F}\ell^1(\mathbb{Z}_+, 1) = W_+$, $F = \mathscr{F}\ell^1(\mathbb{Z}, 1) = W$ (with equivalent norms), and the results are different; we quote them for $w_n = 1$:

(a) $c_n(\delta, A) \ge c_1(\delta, A) = \infty$ for $0 < \delta \le 1/2$ and $c_1(\delta, A) = (1 - 2\delta)^{-1}$ for $1/2 < \delta \le 1$ (see [12]);

(b) $c_n(\delta, F) \le (1 - 2\delta^2)^{-1}$ for all $n \ge 1$ and δ, $1/\sqrt{2} < \delta \le 1$ (and still $c_n(\delta, F) \ge c_1(\delta, F) = \infty$ for $0 < \delta \le 1/2$). By Theorem 1.1, it entails

$$c_n(\delta, W_+) \le 5(1 - 2\delta^4)^{-2} \quad \text{for all } n \ge 2 \text{ and } \delta,\ 1/\sqrt[4]{2} < \delta \le 1,$$

but $c_1(\delta, W_+) = (1 - 2\delta)^{-1}$ for $1/2 < \delta \le 1$.

(2) The case $p > 1$, Sobolev spaces. Sobolev spaces occur for

$$w_n^\alpha = (|n| + 1)^\alpha \quad (n \in \mathbb{Z}).$$

$\mathscr{F}\ell^p(\mathbb{Z}, w_n^\alpha)$ is an algebra if and only if $\alpha > 1/p'$. Inclusion $\mathscr{F}\ell^p(\mathbb{Z}_+, w_n^\alpha) \subset W_+$ means that (here and before) a BS algebra $\mathscr{F}\ell^p(\mathbb{Z}, w_n)$ consists of "smooth" functions on $\mathbb{T}$. All conditions $K_j < \infty$ $(1 \le j \le 6)$ are satisfied, and so $F = \mathscr{F}\ell^p(\mathbb{Z}, w_n^\alpha)$ is a symmetric topological Banach algebra satisfying (A1)–(A4). Theorem 1.1/2.1 is applied with $G = W$; the numerical values of K_j are as follows: $K_3 = K_4 = 1$ (since $w_n^\alpha \uparrow \infty$),

$$K_4 \approx K_2 = \|(1/w_n^\alpha)\|_{\ell^{p'}} \approx (\alpha p' - 1)^{-1/p'}$$

(up to a numerical factor) and a similar value for $K_1 \approx (\alpha p' - 1)^{-1/p'}$ (as it is bounded in [7]). A basic $c_1(\delta, \ell^p(\mathbb{Z}, w_n))$ estimate is given in [7]:

$$c_1(\delta, \ell^p(\mathbb{Z}, w_n)) \le const \cdot \delta^{-\beta},$$

where $\beta = \beta(\alpha, p) > 0$ and the constant also depends on α and p. In [7], this is proved for $\alpha > \frac{1}{2}\left(1 + \frac{1}{p'}\right)$, and in [6] for $p = 1$ and $\alpha > 0$. In both cases, the existing bounds for β are (probably) not optimal: the known values for β vary between

$$\beta = 4\alpha + 2 \quad (\text{for } \alpha > 1 + 1/p')$$

and

$$\beta = \frac{2p'}{\alpha p' - 1} + 2 \quad (\text{for small } \alpha p' - 1 > 0),$$

whereas examples shows that there exists a *const* such that $c_1(\delta, \ell^p(\mathbb{Z}, w_n)) > const \cdot \delta^{-\beta'}$ with $\beta' = \max\big(2, 1 + \frac{\alpha p'-1}{p'}\big)$; see [7].

Notice that an important issue of the power-like growth of $c_1(\delta)$ (as above) is that the topological/Banach algebra passage from §2.2 (replacing $c_1(\delta^2)$ by $c_1(\delta^2/K_1^2)$) is not so painful as for rapidly growing $c_1(\delta)$, because it results in an equivalent majorant (only changing the value of *const*).

4.3. Example: Lipschitz algebras satisfy (A1)–(A4)

Let $0 < \alpha < 1$; the Lipschitz algebra $\mathrm{Lip}(\mathbb{T}, \alpha)$ consists of functions on $\mathbb{T}$ satisfying

$$\|f\|_\alpha = \|f\|_{L^\infty(\mathbb{T})} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

This is a Banach algebra norm ($\|1\|_\alpha = 1$, $\|fg\|_\alpha \le \|f\|_\alpha \|g\|_\alpha$). Properties (A1)–(A3) are classical (for example, $\|z^n\|_\alpha \approx n^\alpha$ implies $R = 1$, whereas the boundedness of $P_+ \colon \mathrm{Lip}(\mathbb{T}, \alpha) \to \mathrm{Lip}(\mathbb{T}, \alpha)$ goes back to I. Privalov (1918), see [5] for references and proofs).

We check condition (A4) with $G = \mathrm{Lip}(\mathbb{T}, \varepsilon)$, $\varepsilon = \alpha/2$, and a constant K_4 having (at most) the order of α^{-2} (up to a numerical constant).

Indeed, we make use the same Hardy-Littlewood equivalent norm which is mentioned in §3.4.4 above,

$$\|f\| = \|f\|_{L^\infty(\mathbb{D})} + \sup_{|z|<1} |f'(z)|(1 - |z|)^{1-\alpha}.$$

Let $\varphi \in \mathrm{Lip}(\mathbb{T}, \alpha)_+$, $0 < r < 1$. Since $|F'(z)| \le \|F\|_\infty (1 - |z|)^{-1}$ for every $F \in H^\infty$, we have (with a norm equivalence constant $c > 0$)

$$\begin{aligned}
\|\varphi'_{r^2}\|_\varepsilon &\le c(\|\varphi'_{r^2}\|_\infty + \sup_{|z|<1} r^2 |\varphi''(r^2 z)|(1 - |z|)^{1-\varepsilon}) \\
&\le c(\|\varphi_{r^2}\|_\infty + \sup_{|z|<1} r \|\varphi'_r\|_\infty (1 - r|z|)^{-1}(1 - |z|)^{1-\varepsilon})
\end{aligned}$$

(and using $(1 - r)^\varepsilon (1 - |z|)^{1-\varepsilon} \le 1 - r|z|$)

$$\begin{aligned}
&\le c(\|\varphi'_{r^2}\|_\infty + r\|\varphi'_r\|_\infty (1 - r)^{-\varepsilon}) \\
&\le c(\|\varphi\|(1 - r)^{\alpha-1} + r\|\varphi\|(1 - r)^{-\varepsilon+\alpha-1}),
\end{aligned}$$

whence $\|\varphi'_{r^2}\|_\varepsilon \le c\|\varphi\|(1-r)^{-1+\alpha/2}$ (with a slightly different constant $c > 0$). Condition (A4) follows (since $\mathrm{Lip}(\mathbb{T}, \varepsilon)$ is a symmetric Banach algebra) with a constant K_4 of the order c/α where $c > 0$ a constant from the Hardy-Littlewood equivalence. The latter also have an order $1/\alpha$ (up to an absolute constant), see the proof of Theorem 5.1 in [5]. The claim follows.

CONCLUSION. *$c_n(\delta, \mathrm{Lip}(\mathbb{T}, \alpha)_+) \le \frac{K}{\delta^8}$ $(0 < \delta \le 1$; $n = 2, 3, \dots)$, where the constant K has (at most) order* $\max\left(\frac{1}{\alpha^3}, \frac{1}{1-\alpha}\right)$, $0 < \alpha < 1$.

Indeed, we use Theorem 1.1, where $G = \mathrm{Lip}(\mathbb{T},\alpha/2)$, $K_2 = K_2(\mathrm{Lip}(\mathbb{T},\alpha) \subset \mathrm{Lip}(\mathbb{T}, \alpha/2)) \le 2^{\alpha/2} \le \sqrt{2}$, $K_3 \le 1$ (see §3.4.5 above), K_4 is of the order $1/\alpha^2$, and $\|P_+\|_{\mathrm{Lip}(\mathbb{T},\alpha)}$ is of the order $\max(1/\alpha, 1/(1-\alpha))$ (for the latter estimate see [1] where the exact value for the norm of Hilbert transform $H = 2P_+ - I$ is found $\|H\|_{\mathrm{Lip}(\mathbb{T},\alpha)} = \pi^{-1}B(\alpha/2, (1-\alpha)/2)$, where B stands for the Euler beta function).

As for the norm controlling constant $c_1(\delta, \mathrm{Lip}(\mathbb{T}, \alpha))$, it can be bounded obviously: $\delta \le |f| \le \|f\|_\alpha \le 1$ implies

$$\|1/f\|_\alpha \le \delta^{-1} + \sup_{x \ne y} \frac{|f(x) - f(y)|}{|f(x)| \cdot |f(y)| \cdot |x - y|^\alpha} \le 2\delta^{-2},$$

so $c_1(\delta, \mathrm{Lip}(\mathbb{T}, \alpha)) \le 2\delta^{-2}$, and the claim follows.

It should be mentioned that some good estimates for Bezout equations in Lipschitz algebras were already obtained in [17] (with a better growth rate $O(1/\delta^2)$ as $\delta \to 0$, but with the use of quite special and complicated techniques of M. Dzhrbashyan integral representations and with no control of the numerical constants, which even may depend on n).

4.4. The Wiener-Dirichlet algebra

This algebra is useful in Toeplitz matrix/operator theory, see [15]. The Dirichlet space $D(\mathbb{T})$ is defined by

$$D(\mathbb{T}) = \mathscr{F}\ell^2(\mathbb{Z}, |n|^{1/2}) = \left\{ f \in L^1(\mathbb{T}) : \|f\|_D^2 := \sum_{n \in \mathbb{Z}} |n| \cdot |\hat{f}(n)|^2 < \infty \right\},$$

and the *Wiener-Dirichlet algebra* by

$$F = WD = W \cap D(\mathbb{T}),$$

equipped with the norm $\|f\| = \|f\|_W + \|f\|_D$. A formula of J. Douglas [3]

$$\|f\|_D^2 = \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right|^2 dm(z_1)\, dm(z_2)$$

shows that $F = WD$ really is a Banach algebra. Also, for $f \in D(\mathbb{T})_+$ (holomorphically extended to the disc $\mathbb{D}$), we obviously have

$$\|f\|_D^2 = \frac{1}{\pi} \int_{\mathbb{D}} |f'(x+iy)|^2 \, dx \, dy.$$

Since $\|S^*\|_{WD_+ \to WD_+} \leq 1$, the algebra $A = WD_+$ is a special case of the backward stable algebras case from §4.1, and hence it satisfies (A1)–(A4) with $G = W$, $K_j = 1$ $(\forall j)$ and $\|P_+\| = 1$.

However, using an (obviously) available estimate $c_1(\delta, WD) \leq c_1(\delta, W) + 1/\delta^2 \leq (2\delta^2 - 1)^{-1} + 2$ (for $1/\sqrt{2} < \delta \leq 1$) and Theorem 1.1, one can only get

$$c_n(\delta, WD_+) \leq 2\big((2\delta^4 - 1)^{-1} + 2\big)^2 \quad (\text{for } 1/\sqrt[4]{2} < \delta \leq 1).$$

The claim follows. It is still unclear whether $c_1(\delta, WD)$ can be infinite for a $\delta > 0$.

REMARK 4.3. A Krein algebra $K = L^\infty D := L^\infty(\mathbb{T}) \cap D(\mathbb{T})$ is even more interesting for Toeplitz analysis, and surely $c_1(\delta, K) < \infty$ for every δ, $0 < \delta \leq 1$, but K is not symmetric in our definition (P_+ is not bounded), and so it does not enter in the above approach to Bezout equations.

Yet another algebra of a similar "mixed nature" is the *Wiener-Lipschitz algebra* $F = W \cap \mathrm{Lip}(\mathbb{T}, \alpha)$, $0 < \alpha \leq 1/2$, endowed with the norm $f \mapsto \|f\| = \|f\|_W + \|f\|_\alpha$ (for $1/2 < \alpha \leq 1$, it is known that $F = \mathrm{Lip}(\mathbb{T}, \alpha)$) for which the functions $\delta \mapsto c_n(\delta, F_+)$ have a similar behaviour to those for the above algebra WD_+. Indeed, as is shown in §3.4.5, one has $M(\mathrm{Lip}(\mathbb{T}, \alpha)_+) = H^\infty$, and hence $\|S^{*n}\| \leq 1$ (on the space $\mathrm{Lip}(\mathbb{T}, \alpha)_+$, and so on F_+); since $F \subset W$, we get (A4) with $G = W$ (in view of §4.1) and can apply Theorem 1.1 with $K_j = 1$ $(\forall j)$, and the claim follows.

5. Proof of Theorem 1.1

The reasoning below represents a simplified form of the classical Uchiyama-Wolff scheme for the proof of the Carleson corona theorem (see for example [11], Appendix 3). However, the most technical arguments (duality and Carleson measure) are omitted and replaced by direct Fourier computations. A similar modification is already used in [8], where the partial case $A = \mathscr{F}\ell^1(\mathbb{Z}_+, w_n)$ is settled.

Let $f = (f_j)_1^n \in A^n$, $0 < \delta \leq |f(z)| \leq \|f\| \leq 1$ $(z \in \mathbb{D})$ and F be a (minimal) symmetric extension of A. We start recalling the Uchiyama-Wolff scheme.

(1) *An approximation*: using (A2) and (A3), we reduce (in a standard way, see for example [11]) the question for real analytic smooth data f, replacing

given f by its Poisson means f_r, then solving $g^{(r)} \cdot f_r = 1$ with $\|g^{(r)}\|_{A^n} \le C$ (as presented below) and finishing with Montel's theorem and (A3) arguments.

From now on, f_j are real analytic, $1 \le j \le n$.

(2) *Anti-symmetric matrix solutions*: setting

$$h_j(z) = \frac{\overline{f}_j(z)}{|f(z)|^2}, \quad z \in \overline{\mathbb{D}},$$

we have a real analytic (and hence, in F) solution to our Bezout equation $h \cdot f = \sum_{j=1}^{n} h_j f_j = 1$, and to correct it up to a solution in A, we set $g = h + H$, where $H \in F^n$ and

$$\overline{\partial} h + \overline{\partial} H = 0 \quad \text{and} \quad H \cdot f = 0,$$

$\overline{\partial} = \frac{\partial}{\partial \overline{z}}$ being the usual complex Cauchy-Riemann derivative. A computational lemma of T. Wolff (see [11]) shows that $\overline{\partial} h + uf = 0$, where u is an anti-symmentic matrix

$$u = (u_{jk})_{1 \le j,k \le n}, \quad u_{jk} = h_j \overline{\partial} h_k - h_k \overline{\partial} h_j,$$

having yet another expression (useful in what follows)

$$u_{jk} = \frac{1}{|f|^4} \left(\overline{f}_j \overline{f}'_k - \overline{f}'_j \overline{f}_k \right).$$

Thus, it remains to find an *anti-symmetric* matrix $v = (v_{jk})_{1 \le j,k \le n}$ ($v_{jk} = -v_{kj}$) solving

$$\overline{\partial} v = u,$$

such that $H := vf \in F^n$. Indeed, in this case, $g = h + vf \in F^n$ and

$$\overline{\partial} g = \overline{\partial} h + \overline{\partial}(vf) = \overline{\partial} h + (\overline{\partial} v) f = \overline{\partial} h + uf = 0 \quad (\Rightarrow \ g \in (H^\infty)^n),$$

so that $g \in A^n$, and since $(vf) \cdot f = 0$ (by anti-symmetry of v),

$$g \cdot f = h \cdot f + (vf) \cdot f = h \cdot f = 1.$$

(3) *A standard Cauchy integral solution*: the equation $\overline{\partial} v = u$ (in $\mathbb{D}$) having a right-hand side real analytic on $\overline{\mathbb{D}}$, has a real analytic solution given by the Cauchy-Green integral

$$v(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{u(\zeta)}{z - \zeta} \, dx \, dy, \quad \zeta = x + iy \quad (z \in \overline{\mathbb{D}}),$$

or – in an entry-by-entry form –

$$v_{jk} = \frac{1}{\pi}\int_{\mathbb{D}} \frac{u_{jk}(\zeta)}{z-\zeta}\,dx\,dy.$$

Clearly, v is anti-symmetric (since $u_{jk} = -u_{kj}$) and anti-analytic on $\mathbb{T}$ (the complex conjugate $\overline{v}$ is in H_0^∞ since $z \mapsto \frac{1}{z-\zeta}$, $|z| = 1$, is $(\forall \zeta \in \mathbb{D})$).

(4) *Bounding* $P_+(vf)$: we have a real analytic function $g = h + vf$ solving $g \cdot f = 1$ and satisfying $\overline{\partial} g = 0$ in $\mathbb{D}$, which implies $g \in A^n$ and $g = P_+ g = P_+ h + P_+(vf)$, and hence

$$\|g\|_{A^n} \le \|P_+ h\|_{A^n} + \|P_+(vf)\|_{A^n}.$$

Since $\delta^2 \le |f|^2 \le \| \, |f|^2\|_F \le \|f\|_{F^n}\|\overline{f}\|_{F^n} = \|f\|_{A^n}^2 \le 1$, we have $\| \, |f|^{-2}\|_F \le c_1(\delta^2, F)$ and

$$\|P_+ h\|_{A^n} \le \|P_+\|_{F\to F}\|\overline{f}\| \cdot \| \, |f|^{-2}\|_F \le c_1(\delta^2, F)\|P_+\|_{F\to F}.$$

Next, setting $V_{kj} = z v_{kj}$,

$$\begin{aligned}\|P_+(vf)\|_{A^n}^2 &= \sum_{k=1}^n \Big\| \sum_{j=1}^n P_+(v_{kj} f_j)\Big\|_A^2 = \sum_{k=1}^n \Big\| \sum_{j=1}^n (v_{kj})_*(S^*) f_j\Big\|_A^2 \\ &\le \sum_{k=1}^n \Big(\sum_{j=1}^n \|S^*\| \cdot \|(V_{kj})_*(S^*)\| \cdot \|f_j\|_A\Big)^2 \\ &\le \|S^*\|^2 \sum_{k=1}^n \Big(\sum_{j=1}^n K_3 \|(V_{kj})_*\|_G \cdot \|f_j\|_A\Big)^2,\end{aligned}$$

where $\|S^*\| = \|S^*\|_{A\to A}$. Now, for $|z| = 1$,

$$\begin{aligned}V_{kj}(z) &= \frac{1}{\pi}\int_{\mathbb{D}} \frac{z u_{kj}(\zeta)}{z-\zeta}\,dx\,dy = \sum_{\ell\ge 0} z^{-\ell}\frac{1}{\pi}\int_{\mathbb{D}} u_{kj}(\zeta)\zeta^\ell\,dx\,dy \\ &= \sum_{\ell \ge 0} z^{-\ell} 2\int_0^1 (u_{kj})_r^\wedge(-\ell) r^{\ell+1}\,dr = 2\int_0^1 r P_-((u_{kj})_r * P_r)(z)\,dr,\end{aligned}$$

where $u_r(e^{it}) = u(re^{it})$ and $P_-\big(\sum_{j\in\mathbb{Z}} a_j z^j\big) = \sum_{j\le 0} a_j z^j$. By the hypothesis of Theorem 1.1, $\|P_-\|_{G\to G} = \|P_+\|_{G\to G} = \|P_+\|_G$ (for brevity) and

$$\begin{aligned}\|(V_{kj})_*\|_G = \|V_{kj}\|_G &\le 2\int_0^1 \|P_-((u_{kj})_r * P_r)\|_G\,dr \\ &\le 2\|P_+\|_G \int_0^1 \|(u_{kj})_r * P_r\|_G\,dr \le 2\|P_+\|_G\int_0^1 \|(u_{kj})_r\|_G\,dr,\end{aligned}$$

The definition of u_{kj} and properties (A1)–(A2) of G and F imply

$$\begin{aligned}\|(u_{kj})_r\|_G &\le \|1/|f_r|^4\|_G\big(\|f_j\|_G\|(f_k')_r\|_G + \|f_k\|_G\|(f_j')_r\|_G\big)\\ &\le (K_2\|1/|f_r|^2\|_F)^2\big(K_2\|f_j\|_A\|(f_k')_r\|_G + K_2\|f_k\|_A\|(f_j')_r\|_G\big)\\ &\le (K_2c_1(\delta^2, F))^2K_2\big(\|f_j\|_A\|(f_k')_r\|_G + \|f_k\|_A\|(f_j')_r\|_G\big).\end{aligned}$$

Integrating and using (A4) we get

$$\|(V_{kj})_*\|_G \le 2\|P_+\|_G K_2^3 c_1(\delta^2, F)^2\big(\|f_j\|_A K_4\|f_k\|_A + \|f_k\|_A K_4\|f_j\|_A\big),$$

so that

$$\begin{aligned}\|P_+(vf)\|_{A^n}^2 &\le K_3^2\|S^*\|^2\sum_{k=1}^n\Big(\sum_{j=1}^n\|(V_{kj})_*\|_G\cdot\|f_j\|_A\Big)^2\\ &\le K_3^2\|S^*\|^2\sum_{k=1}^n\big(2\|P_+\|_G K_2^3c_1(\delta^2, F)^2 2K_4\|f_k\|_A\big)^2,\\ \|P_+(vf)\|_{A^n} &\le 4\|S^*\|K_2^3K_3K_4\|P_+\|_G c_1(\delta^2, F)^2,\end{aligned}$$

and hence

$$\begin{aligned}\|g\|_{A^n} &\le \|P_+h\|_{A^n} + \|P_+(vf)\|_{A^n}\\ &\le c_1(\delta^2, F)\|P_+\|_F + 4\|S^*\|K_2^3K_3K_4\|P_+\|_G c_1(\delta^2, F)^2\\ &\le \big(\|P_+\|_F + 4\|S^*\|K_2^3K_3K_4\|P_+\|_G\big)c_1(\delta^2, F)^2,\end{aligned}$$

and the claim follows.

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