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# ON VECTOR BUNDLES FOR A MORSE DECOMPOSITION OF $L\mathbb{C}\mathrm{P}^n$

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## Abstract

We give a description of the negative bundles for the energy integral on the free loop space $L\mathbb{C}\mathrm{P}^n$ in terms of circle vector bundles over projective Stiefel manifolds. We compute the mod $p$ Chern classes of the associated homotopy orbit bundles.


## 1. Introduction

This paper is a part of a program to study the homotopy type of the free loop space of a smooth manifold $M$. Our main interest is to understand the $\mathbb{T} = S^1$-equivariant homotopy type. More precisely, we try to get information about the mod $p$ equivariant cohomology as a module over the Steenrod algebra.

We remark that this module is closely related to the cohomology of the topological cyclic homology spectrum $TC(M, p)$ [2]. The topological cyclic homology spectrum is in turn an approximation to the algebraic $K$-theory of $M$.

A general strategy for this is to equip the manifold with a Riemannian metric and consider the Morse theory of the energy functional $E$ defined by this metric. Since the energy is invariant under rotation of the loops, this captures not just the ordinary homotopy type of the loop space, but also the equivariant homotopy type.

We focus on a very special case, namely the free loop space on a complex projective space. We choose the Riemannian metric to be the usual (Fubini-Study) metric. We consider this as a special case which might throw light on the general situation.

However, another motivation for examining this special case closely comes from the unsolved closed geodesics problem: does any Riemannian metric on a compact simply-connected smooth manifold $M$ of dimension greater than one admit infinitely many geometrically distinct closed geodesics? The answer is affirmative if the rational cohomology ring of $M$ requires at least

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two generators (Vigué-Poirrier & Sullivan [16], Gromoll & Meyer [5]) or if $M$ is a globally symmetric space of rank larger than one (Ziller [17]). It is also affirmative for the 2-sphere (Bangert, Franks, Angenent, Hingston, see [7]). The most prominent examples where the answer is not known are the spheres $S^m$, $m \geq 3$ together with the projective spaces $\mathbb{C}\mathrm{P}^n$ (for $n \geq 2$), $\mathbb{H}\mathrm{P}^n$ and Cayley's projective plane $\mathbb{O}\mathrm{P}^2$.

In this game, Morse theory of the energy integral on the free loop space $LM$ plays a central role. Therefore it is interesting to gather as much information as possible on the bundles controlling the Morse decomposition.

In [10] Klingenberg studies the non-equivariant Morse theory of the free loop spaces on a projective space $LP^n$. Complex and quaternionic projective spaces as well as the Cayley projective plane are considered. Critical points for the energy integral are closed geodesics of various energy levels $0 = e_0 < e_1 < \cdots$. Those of energy level $e_q$ form a finite-dimensional critical submanifold $B_q$ of $LP^n$. There is a so-called negative vector bundle $\mu_q^-$ over $B_q$ which is essentially the tangent space of the unstable manifold given by exiting negative gradient trajectories. The energy levels also give a filtration of the free loop space $\mathscr{F}(e_q) = E^{-1}([0, e_q])$. Morse theory in this setting states that $\mathscr{F}(e_q)$ is essentially obtained by attaching to $\mathscr{F}(e_{q-1})$ the disc bundle of $\mu_q^-$. One of the results in Klingenberg's article is a concrete calculation of the negative bundles.

By the invariance of the energy functional the filtration is an equivariant filtration. The negative bundles will be $\mathbb{T}$-equivariant bundles, so that they induce vector bundles on the Borel construction on $B_q$. We obtain a filtration of the Borel construction $ES^1 \times_{S^1} LP^n$. The filtration quotients are the Thom spaces of these homotopy orbit bundles over $ES^1 \times_{S^1} B_q$.

The purpose of this paper is firstly to give a simpler description of the negative bundles for the complex projective spaces as $\mathbb{T}$-vector bundles over projective Stiefel manifolds (Theorem 5.10 and Definition 5.8). Secondly, we calculate the mod $p$ Chern classes of the associated homotopy orbit bundles (Theorem 7.10). This determines the action of the Steenrod algebra on the corresponding Thom spaces.

These results are partly motivated by [15] where we compute the mod $p$ equivariant cohomology of $L\mathbb{C}\mathrm{P}^n$ with respect to the action of the circle group $\mathbb{T}$. The calculation uses the spectral sequence coming from the energy filtration. This is a spectral sequence of modules over the Steenrod algebra. The computations in the present paper determine this action on the first page of the spectral sequence, and our hope is that this can lead to a computation of the Steenrod algebra action on $H^*_{\mathbb{T}}(L\mathbb{C}\mathrm{P}^n; \mathbb{F}_p)$.

There is an alternative way of computing equivariant cohomology of $L\mathbb{C}\mathrm{P}^n$. This uses the formality of the homotopy type of $\mathbb{C}\mathrm{P}^n$ together with computations in cyclic homology. The method is described in [14]. At the moment, it does not seem clear how to obtain the action of the Steenrod algebra from this method. However, there is no reason to believe that it is inherently impossible to do this, and our computation might very well help in understanding the relation between cyclic homology and cohomology operations.

## 2. Morse theory for free loop spaces

In this section we recall some results on Morse theory for the energy integral on the Hilbert manifold model of the free loop space. For details we refer to [9].

Let $M$ be a compact Riemannian manifold equipped with the Levi-Civita connection. We use the Hilbert manifold model of the free loop space $LM$. Write the circle as $S^1 = [0, 1]/\{0, 1\}$. An element in $LM$ is an absolutely continuous map $f\colon S^1 \to M$ such that $f'$ is square integrable, i.e. $\int_0^1 |f'(t)|^2\,dt < \infty$. The Hilbert manifold model is homotopy equivalent to the usual continuous mapping space model.

The tangent space $T_f(LM)$ is the set of absolutely continuous tangent vector fields $X$ along $f$ such that the covariant derivative $DX(t)/dt$ is square integrable. The free loop space $LM$ is equipped with a Riemannian metric $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ as follows:

$$\langle\langle X, Y\rangle\rangle = \int_0^1 \left\langle \frac{DX}{dt}(t), \frac{DY}{dt}(t)\right\rangle + \langle X(t), Y(t)\rangle\,dt,$$

where $X, Y \in T_f(LM)$.

The energy integral (or energy function) is defined by

$$E\colon LM \to \mathbb{R};\qquad E(f) = \frac{1}{2}\int_0^1 |f'(t)|^2\,dt.$$

The critical points for $E$ are precisely the closed geodesic on $M$. For a critical point $f$, the Hessian of $E$ has the following form: $H_f(\,\cdot\,,\,\cdot\,)\colon T_f(LM) \times T_f(LM) \to \mathbb{R}$;

$$H_f(X, Y) = \int_0^1 \left\langle \frac{DX}{dt}(t), \frac{DY}{dt}(t)\right\rangle + \big\langle R(X(t), f'(t))f'(t), Y(t)\big\rangle\,dt,$$

where $R(\,\cdot\,,\,\cdot\,)\,\cdot$ denotes the curvature tensor on $M$. The Hessian determines a self-adjoint operator $A_f$ on $T_f(LM)$ satisfying $H_f(X, Y) = \langle\langle A_f(X), Y\rangle\rangle$, for all $X$ and $Y$. The operator $A_f$ is the sum of the identity with a compact operator, so there are at most a finite number of negative eigenvalues, each corresponding to a finite dimensional vector space of eigenvectors of $A_f$. The kernel of $A_f$, which is also finite dimensional, consists of the periodic Jacobi fields along $f$.

Now let $N(e)$ be the space of critical points of $E$ with energy level $e$. It is known that $-\operatorname{grad} E$ satisfies condition (C) of Palais and Smale, so that one can do Morse theory on $LM$ if some additional non-degeneracy condition is satisfied. For us the so-called Bott non-degeneracy condition is the relevant one. It requires firstly that for each critical value $e$ the space $N(e)$ is a compact submanifold of $LM$ and secondly that for each $f \in N(e)$ the restriction of the Hessian $H_f$ to the complement $(T_f N(e))^\perp$ of $T_f N(e)$ in $T_f(LM)$ is non-degenerate. The Bott non-degeneracy condition is a strong assumption on the metric of $M$, but for instance the symmetric spaces satisfy this, according to [17, Theorem 2].

Assume that the Bott non-degeneracy condition holds. The negative bundle $\mu^-(e)$ over $N(e)$ is the vector bundle whose fiber at $f$ is the vector space spanned by the eigenvectors belonging to negative eigenvalues of $A_f$. Similarly, $\mu^0(e)$ and $\mu^+(e)$ are the vector bundles with fibers spanned by the eigenvectors corresponding to the eigenvalue 0 and the positive eigenvalues respectively.

Let the critical values of the energy function be $0 = e_0 < e_1 < \cdots$. Consider the filtration of $LM$ given by $\mathscr{F}(e_i) = E^{-1}([0, e_i])$. This filtration is equivariant with respect to the action of the circle.

The tangent bundle of $LM$ restricted to $N(e_i)$ splits $\mathbb{T}$-equivariantly into a sum of three bundles.

$$T(LM)|_{N(e_i)} \cong \mu^-(e_i) \oplus \mu^0(e_i) \oplus \mu^+(e_i).$$

The standard Morse theory argument can be carried through equivariantly on the Hilbert manifold $LM$. This was done by Klingenberg. For an account of this work see section [11, 2.4], especially Theorem 2.4.10. The statement of this theorem implies that we have an equivariant homotopy equivalence

$$\mathscr{F}(e_i)/\mathscr{F}(e_{i-1}) \simeq \mathrm{Th}(\mu^-(e_i)).$$

## 3. Klingenberg's calculation of negative bundles for projective spaces

We will now focus on the projective spaces $P^n(\alpha)$ over the complex numbers $\mathbb{C}$ for $\alpha = 2$, the quaternions $\mathbb{H}$ for $\alpha = 4$ and the Cayley numbers $\mathbb{O}$ for $\alpha = 8$. Note that $P^n(8)$ only exist when $n = 1$ or $n = 2$. These spaces are endowed with the Riemannian metric which makes them symmetric spaces of rank one. This metric is determined up to a positive constant, which we fix by requiring the sectional curvature to have maximal value $2\pi^2$ and minimal value $\pi^2/2$ [10, 1.1].

Klingenberg calculates the negative bundles for $L(P^n(\alpha))$ in [10] and we will review this calculation.

Let $B_q(P^n(\alpha)) \subseteq LP^n(\alpha)$ denote the critical submanifold of $q$-fold covered primitive geodesics. A non-constant geodesic $f \in B_q(P^n(\alpha))$ lies on a unique projective line $S^\alpha \cong P^1(\alpha) \subseteq P^n(\alpha)$. For each $t \in [0, 1]$, we split the tangent space at $f(t)$ into a horizontal subspace of tangent vectors to this projective line and its orthogonal complement, called the vertical subspace [10, 1.3],

$$T_{f(t)}(P^n(\alpha)) = T_{f(t)}(P^n(\alpha))_h \oplus T_{f(t)}(P^n(\alpha))_v.$$

The horizontal subspace has real dimension $\alpha$ and the vertical subspace has real dimension $\alpha(n-1)$. A tangent vector field $X \in T_f(P^n(\alpha))$ decomposes into a horizontal component $X_h$ and a vertical component $X_v$ and this decomposition is compatible with the covariant derivative along $f$.

PROPOSITION 3.1 (Klingenberg). *Consider the parallel transport around a simple closed geodesic $f\colon [0, 1] \to P^n(\alpha)$ with $f(0) = f(1) = p$. The horizontal subspace of $T_p(P^n(\alpha))$ is carried into itself by the identity map. The vertical subspace is carried into itself by the reflection at the origin.*

We will not review Klingenberg's proof here. A proof for the complex projective space will however appear later in Lemma 5.1.

LEMMA 3.2 (Klingenberg). *Let $f \in B_q(P^n(\alpha))$, where $q$ is a positive integer. The Hessian $H_f(\cdot, \cdot)$ on $T_f(LP^n(\alpha))$ has eigenvectors as follows:*

(1)
$$X_p(t) = A\cos(2\pi pt) + B\sin(2\pi pt), \quad p \in \mathbb{N}_0,$$

*where $A$ and $B$ are constant (i.e. parallel) horizontal vector fields along $f$ such that $\langle A, f'(t)\rangle = \langle B, f'(t)\rangle = 0$ for all $t$. The eigenvalue for $X_p$ is*
$$\lambda_p = \frac{4\pi^2(p^2 - q^2)}{1 + 4\pi^2 p^2}.$$

*We write $E_{h,p}$ for the vector space formed by the $X_p$'s for a fixed $p$. It has real dimension $\alpha - 1$ for $p = 0$ and $2(\alpha - 1)$ for $p > 0$.*

(2)
$$Y_r(t) = A\cos(\pi rt) + B\sin(\pi rt), \quad r \in \mathbb{N}_0,\ r \equiv q \bmod 2,$$

*where $A$ and $B$ are constant vertical vector fields along $f$. The eigenvalue of $Y_r$ is*
$$\mu_r = \frac{\pi^2(r^2 - q^2)}{1 + \pi^2 r^2}.$$

*We write $E_{v,r}$ for the vector space formed by the $Y_r$'s. It has real dimension $\alpha(n-1)$ if $r = 0$ and $2\alpha(n-1)$ if $r > 0$.*

(3)
$$Z_s(t) = (a\cos(2\pi st) + b\sin(2\pi st))f'(t), \quad s \in \mathbb{N}_0,$$

*where $a, b \in \mathbb{R}$. The eigenvalue for $Z_s$ is*

$$\nu_s = \frac{4\pi^2 s^2}{1 + 4\pi^2 s^2}.$$

*We write $E_{t,s}$ for the vector space formed by the $Z_s$'s. It has real dimension 1 for $s = 0$ and 2 for $s > 0$.*

Proof. The proposition above and the parity condition in (2) ensures that $X_p(0) = X_p(1)$ and $Y_r(0) = Y_r(1)$.

With our choice of metric, $|f'(t)|^2 = 2q^2$. Moreover, the curvature tensor for $P^n(\alpha)$ is known, and its block matrix form allows Klingenberg to decompose the Hessian into a horizontal and a vertical quadratic form [10, 1.4]

$$\begin{aligned} H_f^h(X_h, Y_h) = \int_0^1 & \left\langle \frac{DX_h}{dt}(t), \frac{DY_h}{dt}(t) \right\rangle \\ & - 2\pi^2(2q^2\langle X_h(t), Y_h(t)\rangle - \langle f'(t), X_h(t)\rangle\langle f'(t), Y_h(t)\rangle)\, dt, \\ H_f^v(X_v, Y_v) = \int_0^1 & \left\langle \frac{DX_v}{dt}(t), \frac{DY_v}{dt}(t) \right\rangle - \pi^2 q^2 \langle X_v(t), Y_v(t)\rangle\, dt. \end{aligned}$$

Consider the eigen-equation $H_f^h(X_h, Y_h) = \lambda\langle\langle X_h, Y_h\rangle\rangle$ for all $Y_h$, where $\lambda \in \mathbb{R}$. If $X_h$ possess a second covariant derivative, we get an equivalent equation via partial integration

$$(1-\lambda)\frac{D^2 X_h}{dt^2} + (4\pi^2 q^2 + \lambda)X_h - 2\pi^2\langle f', X_h\rangle f' = 0. \tag{1}$$

We insert $X_p$ in this equation. Since $D^2X_p/dt^2 = -4\pi^2 p^2 X_p$, we get the following:

$$((4\pi^2 p^2 + 1)\lambda - 4\pi^2(p^2 - q^2))X_p = 0.$$

Thus $\lambda_p$ is an eigenvalue for $H_f^h(\cdot\,,\,\cdot)$ with eigenvector $X_p$.

From $H_f^v(X_v, Y_v) = \mu\langle\langle X_v, Y_v\rangle\rangle$ for all $Y_v$, where $\mu \in \mathbb{R}$, we get the eigen-equation

$$(1-\mu)\frac{D^2 X_v}{dt^2} + (\pi q^2 + \mu)X_v = 0.$$

We insert $Y_r$. Since $D^2Y_r/dt^2 = -\pi^2 r^2 Y_r$, we get

$$((\pi^2 r^2 + 1)\mu - \pi^2(r^2 - q^2))Y_r = 0.$$

Thus $\mu_r$ is an eigenvalue for $H_f^v(\cdot\,,\,\cdot)$ with eigenvector $Y_r$.

Finally, we insert $Z_s$ into (1). Since $f$ is a geodesic we have that $Df/dt = 0$. Thus, $D^2Z_s/dt^2 = -4\pi^2 s^2 Z_s$ and we obtain

$$((1 + 4\pi^2 s^2)\lambda - 4\pi^2 s^2)Z_s = 0.$$

We see that $\nu_s$ is an eigenvalue for $H_f^h(\cdot\,,\,\cdot)$ with eigenvector $Z_s$.

The subspaces described in (1)–(3) have trivial pairwise intersection. They also generate the full Hilbert space $T_f(P^n(\alpha))$, so we have the following result:

COROLLARY 3.3. *The negative subspace is the direct sum*

$$T_f(LP^n(\alpha))^- = \bigoplus_{0\le p<q} E_{h,p} \oplus \bigoplus_{0\le r<q,\ r\equiv q \bmod 2} E_{v,r}.$$

*It has real dimension* $(2q-1)(\alpha-1) + (q-1)\alpha(n-1)$.

*The zero subspace is*

$$T_f(LP^n(\alpha))^0 = E_{t,0} \oplus E_{h,q} \oplus E_{v,q}.$$

*It has real dimension* $2\alpha n - 1$.

*The positive subspace is the Hilbert direct sum*

$$T_f(LP^n(\alpha))^+ = \bigoplus_{p>q} E_{h,p} \oplus \bigoplus_{r>q,\ r\equiv q \bmod 2} E_{v,r} \oplus \bigoplus_{s>0} E_{t,s}.$$

Klingenberg shows that there are vector bundles over $B_q(P^n(\alpha))$, for $q \ge 1$, as follows:

| Vector bundle | $\dim_{\mathbb{R}}$ | Fiber over $f$ | Condition |
|---|---|---|---|
| $\eta_{h,0}$ | $\alpha - 1$ | $E_{h,0}$ | |
| $\sigma_{h,p}$ | $2(\alpha - 1)$ | $E_{h,p}$ | $p \ge 1$ |
| $\sigma_{v,2p-1}$ | $2\alpha(n-1)$ | $E_{v,2p-1}$ | $q$ odd, $p \ge 1$ |
| $\eta_{v,0}$ | $\alpha(n-1)$ | $E_{v,0}$ | $q$ even |
| $\sigma_{v,2p}$ | $2\alpha(n-1)$ | $E_{v,2p}$ | $q$ even |

Thus, we have the following result [10, 1.6]:

THEOREM 3.4 (Klingenberg). *The non-trivial critical points for the energy integral $E\colon L(P^n(\alpha)) \to \mathbb{R}$ decompose into the non-degenerate critical submanifolds $B_q(\alpha) = B_q(P^n(\alpha))$ consisting of the $q$-fold covered parametrized great circles, $q = 1, 2, \ldots$; $E(B_q(\alpha)) = 2q^2$. The negative bundle $\mu_q^-$ over $B_q(\alpha)$ has the following form:*

$$\mu_q^- = \begin{cases} \eta_{h,0} \oplus \bigoplus_{p=1}^{q-1} \sigma_{h,p} \oplus \bigoplus_{p=1}^{(q-1)/2} \sigma_{v,2p-1}, & \text{for } q \text{ odd,} \\ \eta_{h,0} \oplus \bigoplus_{p=1}^{q-1} \sigma_{h,p} \oplus \eta_{v,0} \oplus \bigoplus_{p=1}^{(q-2)/2} \sigma_{v,2p}, & \text{for } q \text{ even.} \end{cases}$$

## 4. Spaces of geodesics viewed as projective Stiefel manifolds

From now on, we consider the complex projective space $\mathbb{C}\mathrm{P}^n$. It has a Hermitian metric, which we now describe. References are [12], page 273, or [13], page 142.

Equip $\mathbb{C}^{n+1}$ with the standard Hermitian inner product $h(v, w) = \sum_{k=1}^{n+1} v_k \overline{w}_k$. The real part $g'(v, w) = \operatorname{Re} h(v, w)$ is the usual inner product on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. Furthermore, $h(v, w) = g'(v, w) + ig'(v, iw)$.

Let $S^{2n+1} = \{x \in \mathbb{C}^{n+1} \mid h(x, x) = 1\}$ be the unit sphere and write $\mathbb{T}$ for the unit circle group. Consider the Hopf projection

$$\rho\colon S^{2n+1} \to S^{2n+1}/\mathbb{T} = \mathbb{C}\mathrm{P}^n.$$

By restriction of $h$, we have a Hermitian inner product on the orthogonal complement $(\mathbb{C}x)^\perp = \{v \in \mathbb{C}^{n+1} \mid h(x, v) = 0\}$ and $(\mathbb{C}x)^\perp$ is a real subspace of the tangent space $T_x(S^{2n+1})$. One can equip $\mathbb{C}\mathrm{P}^n$ with a Hermitian metric $\tilde{h}(\cdot\,, \cdot)$ such that

$$d\rho_x\colon (\mathbb{C}x)^\perp \subseteq T_x(S^{2n+1}) \xrightarrow{\rho_*} T_{\rho(x)}(\mathbb{C}\mathrm{P}^n)$$

becomes a $\mathbb{C}$-linear isometry. The following identity holds

$$d\rho_{zx}(zv) = d\rho_x(v), \quad \text{for } z \in \mathbb{T}. \tag{2}$$

The real part $\tilde{g}(\cdot\,, \cdot) = \operatorname{Re} \tilde{h}(\cdot\,, \cdot)$ is the Fubini-Study metric on $\mathbb{C}\mathrm{P}^n$. (In [12], they allow a rescaling of $\tilde{g}$ by $4/c$ for a positive constant $c$. We let $c = 4$.) It is known that the sectional curvature for this metric has maximal value 4 and minimal value 1 when $n > 1$. Thus the metric on $\mathbb{C}\mathrm{P}^n$ used in Section 3 is $\frac{\pi^2}{2}\tilde{g}$.

For $\mathbb{C}\mathrm{P}^n$ with Riemannian metric $\tilde{g}$ and associated Levi-Civita connection, we now describe the spaces of closed geodesics $B_q(\mathbb{C}\mathrm{P}^n)$ in terms of projective

Stiefel manifolds. Recall that $B_q(\mathbb{C}\mathrm{P}^n)$ is the space of constant geodesics for $q = 0$, primitive geodesics for $q = 1$ and $q$-fold iterated primitive geodesics for $q \geq 2$.

DEFINITION 4.1. Let $\mathbf{V}_2(\mathbb{C}^{n+1})$ denote the Stiefel manifold of complex orthonormal 2-frames in $\mathbb{C}^{n+1}$.

Write $U$ for the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

and let $D_\theta$ and $R_\theta$ be the following diagonal and rotation matrices:

$$D_\theta = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}, \quad R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

LEMMA 4.2. *Matrix multiplication defines a right action*

$$\mathbf{V}_2(\mathbb{C}^{n+1}) \times U(2) \to \mathbf{V}_2(\mathbb{C}^{n+1});$$
$$\left((u, v), \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \mapsto (au + cv, bu + dv).$$

*The diffeomorphism* $\tau\colon \mathbf{V}_2(\mathbb{C}^{n+1}) \to \mathbf{V}_2(\mathbb{C}^{n+1})$; $(u, v) \mapsto (u, v)U$ *satisfies*

$$\tau((u, v)D_\theta) = \tau(u, v)R_\theta.$$

PROOF. Regarding the action, it suffices to verify that the image frame is orthonormal. By the elementary properties of the inner product, one finds that

$$h(au + cv, au + cv) = 1,$$
$$h(au + cv, bu + dv) = 0,$$
$$h(bu + dv, bu + dv) = 1.$$

so this is the case. Let

$$V = U^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}.$$

One has

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (\alpha - i\beta) \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = (\alpha + i\beta) \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

For $\alpha = \cos\theta$ and $\beta = \sin\theta$ this gives us the diagonalization $V^{-1}R_\theta V = D_\theta$. Thus, $UR_\theta = D_\theta U$ such that $\tau$ has the stated property.

We now define a right action of the torus group $\mathbb{T}^2$ on the Stiefel manifold. We use different notations for the left and right circle group factors as follows: $\mathbb{T}^2 = \mathbb{T} \times U(1)$. We view $\mathbb{T}$ and $U(1)$ as subgroups of the abelian group $\mathbb{T}^2$ via inclusion in the first and second factor respectively. For each integer $q$ there is a group homomorphism

$$\iota_q : \mathbb{T}^2 \to U(2); \quad (z_1, z_2) \mapsto \begin{bmatrix} z_1^q z_2 & 0 \\ 0 & z_2 \end{bmatrix}.$$

Recall that a right $G$-space $X$ is considered a left $G$-space by the action $g * x = x * g^{-1}$ for $g \in G$, $x \in X$ and vice versa.

DEFINITION 4.3. The torus $\mathbb{T}^2$ acts from the right on $\mathbf{V}_2(\mathbb{C}^{n+1})$ via the homomorphism $\iota_q$ and the $U(2)$-action of Lemma 4.2. Let $\mathbf{V}_{2,q}(\mathbb{C}^{n+1})$ denote the corresponding left $\mathbb{T}^2$-space. The projective Stiefel manifold is defined as the quotient space

$$\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) = \mathbf{V}_{2,q}(\mathbb{C}^{n+1})/U(1).$$

It is equipped with a left action of the quotient group $\mathbb{T} \cong \mathbb{T}^2/U(1)$. When viewed as a space without a group action, the projective Stiefel manifold is denoted $\mathbf{PV}_2(\mathbb{C}^{n+1})$.

REMARK 4.4. Alternatively, we have

$$\mathbf{PV}_2(\mathbb{C}^{n+1}) = \mathbf{V}_2(\mathbb{C}^{n+1})/\operatorname{diag}_2(U(1))$$

where $\operatorname{diag}_2(U(1)) \subseteq U(2)$ denotes the diagonal inclusion. The $\mathbb{T}$-action is given by

$$z * [u, v] = [z^{-q}u, v] = [u, z^q v] = [c^{-q}u, c^q v],$$

where $c$ is a square root of $z$. Note that $[u, zv] = [u, v] \Rightarrow z = 1$, so the $\mathbb{T}$-action is free when $q = 1$.

The projective Stiefel manifold is diffeomorphic to the sphere bundle of the tangent bundle of $\mathbb{C}\mathrm{P}^n$ as follows:

$$\Phi : \mathbf{PV}_2(\mathbb{C}^{n+1}) \xrightarrow{\cong} S(T(\mathbb{C}\mathrm{P}^n)); \quad [u, v] \longmapsto (d\rho_u(v))_{\rho(u)}.$$

So via the exponential map it corresponds to a space of geodesics. The $\mathbb{T}$-action on $\mathbf{PV}_{2,1}(\mathbb{C}^{n+1})$ corresponds to complex rotation in the tangent bundle since $\Phi([u, zv]) = (zd\rho_u(v))_{\rho(u)}$. The purpose of the diffeomorphism $\tau$ of

Lemma 4.2 is to make this $\mathbb{T}$-action, which has a simple description, correspond to rotation of closed geodesics. More precisely we have:

THEOREM 4.5. *For every positive integer $q$ there is a $\mathbb{T}$-equivariant diffeomorphism*

$$\phi_q\colon \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^n); \quad \phi_q([u,v])(t) = \rho\left(\frac{e^{-q\pi i t}u + e^{q\pi i t}v}{\sqrt{2}}\right).$$

PROOF. It is well known ([4], 2.110, or [12], page 277) that there is a diffeomorphism

$$\psi_q\colon \mathbf{PV}_2(\mathbb{C}^{n+1}) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^n);$$
$$\psi_q([a,b])(t) = \rho\big(\cos(q\pi t)a + \sin(q\pi t)b\big) = \rho\left((a,b)R_{q\pi t}\begin{bmatrix}1\\0\end{bmatrix}\right),$$

where $0 \le t \le 1$. The diffeomorphism becomes equivariant when we let $\mathbb{T}$ act on $\mathbf{B}_q(\mathbb{C}\mathrm{P}^n)$ and $\mathbf{PV}_2(\mathbb{C}^{n+1})$ by $(e^{2\pi i s} * f)(t) = f(s+t)$ and $e^{2\pi i s} \star [a,b] = [(a,b)R_{q\pi s}]$ respectively. Write $\mathbf{PV}_{2,(q)}(\mathbb{C}^{n+1})$ for the projective Stiefel manifold equipped with this action.

The group $\mathrm{diag}_2(U(1))$ is in the center of $U(2)$ so the map $\tau$ from Lemma 4.2 gives us a well-defined automorphism of the projective Stiefel manifold. This automorphism is a $\mathbb{T}$-equivariant map

$$\tau_q\colon \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \to \mathbf{PV}_{2,(q)}(\mathbb{C}^{n+1})$$

by the equation for $\tau$ proven in Lemma 4.2. Via Euler's formulas we find

$$(\psi_q \circ \tau_q)([u,v])(t) = \rho\left(\frac{e^{-iq\pi t}u + e^{iq\pi t}v}{\sqrt{2}}\right).$$

Thus, $\psi_q \circ \tau_q = \phi_q$ and we have the desired result.

## 5. A description of the negative bundle

In this section we will describe the negative bundles as bundles over projective Stiefel manifolds. We start with the following result regarding the constant (parallel) horizontal and vertical vector fields mentioned in Lemma 3.2.

LEMMA 5.1. *Let $(u,v) \in \mathbf{V}_2(\mathbb{C}^{n+1})$ and let $q$ be a positive integer. Define the curve*

$$c\colon [0,1] \to S^{2n+1}; \quad c(t) = \frac{e^{-q\pi i t}u + e^{q\pi i t}v}{\sqrt{2}}$$

*and put* $f(t) = \rho(c(t)) = \phi_q([u, v])(t)$. *Then the horizontal and vertical subspaces at* $f(t)$ *are given by*

$$T_{f(t)}(\mathbb{C}\mathrm{P}^n)_h = d\rho_{c(t)}(\mathrm{span}_{\mathbb{C}}(c'(t))), \quad T_{f(t)}(\mathbb{C}\mathrm{P}^n)_v = d\rho_{c(t)}(\{u, v\}^{\perp}),$$

*where* $\perp$ *is with respect to the Hermitian inner product* $h$. *Furthermore,*

$$H(t) = d\rho_{c(t)}(e^{-q\pi it}u - e^{q\pi it}v)$$

*is a parallel and horizontal vector field along* $f$, *such that* $\tilde{g}(H(t), f'(t)) = 0$ *for all* $t$, *and*

$$V(w)(t) = d\rho_{c(t)}(w)$$

*is a parallel and vertical vector field along* $f$ *for all* $w \in \{u, v\}^{\perp}$. *These vector fields satisfy*

$$H(0) = H(1), \quad V(w)(0) = (-1)^q V(w)(1).$$

Proof. We have that $c'(t) = -q\pi i(e^{-q\pi it}u - e^{q\pi it}v)/\sqrt{2}$. Since $u$ and $v$ are orthonormal vectors it follows that $h(c'(t), c'(t)) = q^2\pi^2$ and $h(c(t), c'(t)) = 0$. Furthermore, $\{c(t), c'(t)\}^{\perp} = \{u, v\}^{\perp}$, for all $t$. Thus we have an orthogonal decomposition

$$\{c(t)\}^{\perp} = \mathrm{span}_{\mathbb{C}}(c'(t)) \oplus \{c(t), c'(t)\}^{\perp} = \mathrm{span}_{\mathbb{C}}(c'(t)) \oplus \{u, v\}^{\perp}.$$

By the chain rule, $f'(t) = T_{c(t)}(\rho)(c'(t)) = d\rho_{c(t)}(c'(t))$ so that

$$T_{f(t)}(\mathbb{C}\mathrm{P}^n)_h = \mathrm{span}_{\mathbb{C}}(f'(t)) = \eta_{c(t)}(\mathrm{span}_{\mathbb{C}}(c'(t)))$$

and, since $d\rho_{c(t)}$ is an isometry, we also obtain the desired descriptions of the vertical subspace.

Put $\tilde{H}(t) = e^{-q\pi it}u - e^{q\pi it}v$. Since $\tilde{H}$ is a complex rescaling of $c'$, we see that $H$ is a horizontal vector field.

We have equipped $S^{2n+1} \subseteq \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ with the Riemannian metric induced from $\mathbb{R}^{2n+2}$. Since $c$ is a geodesics in that metric we have $D\tilde{H}(t)/dt = 0$. The projective space $\mathbb{C}\mathrm{P}^n$ is equipped with the Fubini-Study metric so it follows that $DH(t)/dt = 0$. Thus $H$ is a parallel vector field along $f$.

We have $h(\tilde{H}(t), c'(t)) = -q\pi i\|\tilde{H}(t)\|^2/\sqrt{2}$. The real part of this equation gives us that $g'(\tilde{H}(t), c'(t)) = 0$. It follows that $\tilde{g}(H(t), f'(t)) = 0$, since $d\rho_{c(t)}$ is an isometry.

By the first part of the lemma, $V(w)$ is a vertical vector field for all $w \in \{u, v\}^{\perp}$. Since $w$ is constant, $dw/dt = 0$. So its orthogonal projection $Dw/dt$ onto the tangent space at $c(t)$ is also zero. It follows that $DV(w)/dt = 0$

such that $V(w)$ is a parallel vector field along $f$. The final relations follows by equation (2).

We will now give a slightly different description of the curve and vector fields of the lemma such that the proof of Theorem 5.9 becomes easier.

DEFINITION 5.2. For $(u, v) \in \mathbf{V}_2(\mathbb{C}^{n+1})$, we define the *closed* geodesic

$$c(u, v)\colon \mathbb{T} \to S^{2n+1}; \quad c(u, v)(z) = \frac{1}{\sqrt{2}}(z^{-1}u + zv).$$

The equivariant diffeomorphism $\phi_q\colon \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^n)$ from Theorem 4.5 is defined by the diagram

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "c(u,v)"] & S^{2n+1} \arrow[d, "\rho"] \\
\mathbb{T} \arrow[u, "(\sqrt{\cdot}\,)^q"] \arrow[r, "\phi_q([u,v])"] & \mathbb{C}\mathrm{P}^n.
\end{tikzcd}$$

Note that $h(c(u, v), c(u, -v)) = 0$. So we can view $c(u, -v)$ as a vector field along $c(u, v)$.

DEFINITION 5.3. Define a parallel horizontal tangent vector field along $\phi_2([u, v])$ by

$$H(u, v)(z) = d\rho_{c(u,v)(z)}\big(c(u, -v)(z)\big)$$

and for $w \in \{u, v\}^\perp$, where $\perp$ is with respect to $h$, a parallel vertical tangent vectors field by

$$V(u, v, w)(z) = d\rho_{c(u,v)(z)}(w).$$

The relations to the curve and vector fields of Lemma 5.1 are as follows:

$$\begin{aligned}
c(u, v)(e^{q\pi it}) &= c(t), \\
H(u, v)(e^{q\pi it}) &= H(t), \\
V(u, v, w)(e^{q\pi it}) &= V(w)(t).
\end{aligned}$$

PROPOSITION 5.4. *For all $\lambda \in U(1)$, one has the identities*

$$H(\lambda u, \lambda v) = H(u, v), \quad V(\lambda u, \lambda v, \lambda w) = V(u, v, w).$$

*Furthermore, for all $z_1, z_2 \in \mathbb{T}$, one has*

$$H(u, v)(z_1 z_2) = H(u, z_1^2 v)(z_2), \quad V(u, v, w)(z_1 z_2) = V(u, z_1^2 v, z_1 w)(z_2).$$

*As special cases,*

$$H(u,v)(-z) = H(u,v)(z) \quad \text{and} \quad V(u,v,w)(-z) = V(u,v,-w)(z).$$

PROOF. The first two identities follow using equation (2). From Definition 5.2, one sees that

$$c(u,v)(z_1z_2) = c(z_1^{-1}u, z_1v)(z_2)$$

This relation and the first two identities gives the last two identities.

We now have sufficient information on the constant horizontal and vertical vector fields in Klingenberg's Lemma 3.2. Next we will define the bundles over projective Stiefel manifolds which correspond to the summands of the negative bundle.

The concept of $G$-vector bundle (over the real or complex numbers), for a topological group $G$, will be used ([1], §1.6). A $G$-space $E$ is a $G$-vector bundle over a $G$-space $X$ if

(i) $E$ is a vector bundle over $X$,

(ii) the projection $E \to X$ is a $G$-map,

(iii) for each $g \in G$ the map $g\cdot\colon E_x \to E_{gx}$ is a vector space homomorphism.

In the special case where the action of $G$ on $X$ is trivial, we see that each fiber becomes a $G$-module.

PROPOSITION 5.5. *Let $G$ be a compact Lie group with a closed normal subgroup $H \subseteq G$. Let $X$ be a $G$-space such that the canonical projection $X \to X/H$ is a principal $H$-bundle.*

(1) *If $\eta \to X$ is a $G$-vector bundle then $\eta/H \to X/H$ is a $G/H$-vector bundle.*

(2) *For $G$-vector bundles $\eta_1 \to X$ and $\eta_2 \to X$ there is a natural isomorphism of $G/H$-vector bundles*

$$(\eta_1 \oplus \eta_2)/H \cong \eta_1/H \oplus \eta_2/H.$$

(3) *If $\xi_1 \to Y$ and $\xi_2 \to Y$ are $G$-vector bundles and $f\colon X \to Y$ is a $G$-map then there is a natural isomorphism of $G/H$-vector bundles*

$$f^*(\xi_1 \oplus \xi_2)/H \cong f^*(\xi_1)/H \oplus f^*(\xi_2)/H.$$

PROOF. (1) Let $p: E \to X$ be the projection map for $\eta$. By [3], I.3.4, there is a $G/H$-action on $E/H$ such that the following diagram commutes:

$$\begin{tikzcd}
G \times E \arrow[r] \arrow[d] & E \arrow[d] \\
G/H \times E/H \arrow[r] & E/H.
\end{tikzcd}$$

Likewise, we have a $G/H$-action on $X/H$ and $p/H$ is a $G/H$-map by naturality. Thus, condition (ii) holds.

Furthermore, $p/H: E/H \to X/H$ is a vector bundle by [3], I.9.4, such that (i) holds, and there is a pullback diagram of vector bundles

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "p"] & E/H \arrow[d, "p/H"] \\
X \arrow[r] & X/H.
\end{tikzcd}$$

Finally, the first of the diagrams above gives us a commutative diagram of fibers for $x \in X$ and $g \in G$:

$$\begin{tikzcd}
E_x \arrow[r, "g\cdot"] \arrow[d, "\cong"] & E_{gx} \arrow[d, "\cong"] \\
(E/H)_{[x]} \arrow[r, "{[g]\cdot}"] & (E/H)_{[gx]}.
\end{tikzcd}$$

The top map is linear since $E \to X$ is a $G$-vector bundle. The vertical maps are isomorphisms by the pullback diagram above. So condition (iii) also holds.

(2) There is a well-defined map $\psi$ which makes the following diagram commute:

$$\begin{tikzcd}
\eta_1 \oplus \eta_2 \arrow[r] \arrow[d, equal] & (\eta_1 \oplus \eta_2)/H \arrow[d, "\psi"] \\
\eta_1 \oplus \eta_2 \arrow[r] & \eta_1/H \oplus \eta_2/H.
\end{tikzcd}$$

The bottom map is surjective, so $\psi$ is also surjective. Furthermore, $\psi$ is a bundle map over $X/H$ which maps a fiber of its domain to an isomorphic fiber of its codomain by the pullback diagram above. So $\psi$ is an isomorphism of vector bundles. One sees directly by its transformation rule $\psi([v_1, v_2]) = ([v_1], [v_2])$ that $\psi$ is a $G/H$-map.

(3) The standard isomorphism $f^*(\xi_1 \oplus \xi_2) \cong f^*(\xi_1) \oplus f^*(\xi_2)$ is $G$-equivariant, so we have an isomorphism $f^*(\xi_1 \oplus \xi_2)/H \cong (f^*(\xi_1) \oplus f^*(\xi_2))/H$ of $G/H$-vector bundles. The result then follows by (2).

The projection map $\mathbf{V}_2(\mathbb{C}^{n+1}) \to \mathbf{V}_2(\mathbb{C}^{n+1})/\operatorname{diag}_2(U(1)) = \mathbf{PV}_2(\mathbb{C}^{n+1})$ is a principal $U(1)$-bundle by standard arguments. So by (1) in the proposition above, we have the following construction of $\mathbb{T}$-vector bundles:

Definition 5.6. Let $f: \mathbf{V}_{2,q}(\mathbb{C}^{n+1}) \to X$ be a $\mathbb{T}^2$-map and let $\xi$ be a complex $\mathbb{T}^2$-vector bundle over $X$. Form the pullback $f^*(\xi)$. The quotient $f^*(\xi)/U(1)$ is a complex $\mathbb{T}$-vector bundle which we denote

$$\mathbf{PV}_{2,q}(f,\xi) \to \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}).$$

We only need this construction for a special type of torus vector bundle.

Definition 5.7. Let $\eta \to X$ be a complex vector bundle and $i$, $j$ two integers. Equip the total space of $\eta$ with a $\mathbb{T}^2$-action via complex multiplication in the fibers as follows:

$$(z_1, z_2) * v = z_1^i z_2^j v.$$

The resulting $\mathbb{T}^2$-vector bundle over the trivial $\mathbb{T}^2$-space $X$ is denoted $\eta(i, j)$.

Let $\gamma_2$ be the canonical bundle over the Grassmannian $\mathbf{G}_2(\mathbb{C}^{n+1})$. Its total space consists of the pairs $(V, v)$, where $V$ is a complex two-dimensional subspace of $\mathbb{C}^{n+1}$ and $v \in V$. It has an orthogonal complement bundle $\gamma_2^\perp$ over $\mathbf{G}_2(\mathbb{C}^{n+1})$ consisting of pairs $(V, w)$, where $w \in V^\perp \subseteq \mathbb{C}^{n+1}$. Let $\pi: \mathbf{V}_{2,q}(\mathbb{C}^{n+1}) \to \mathbf{G}_2(\mathbb{C}^{n+1})$ be the projection which maps a frame to its complex span. We equip the Grassmannian with the trivial $\mathbb{T}^2$-action, so that $\pi$ becomes equivariant. Finally, for a complex vector space $V$, we write $\overline{V}$ for its conjugate vector space. As real vector spaces $V$ and $\overline{V}$ are the same but $z \cdot v = \bar{z}v$, for $v \in V$ and $z \in \mathbb{C}$. For a complex vector bundle $\xi$, we write $\overline{\xi}$ for its conjugate vector bundle.

Definition 5.8. For $r = q \bmod 2$ we define $\mathbb{T}$-vector bundles as follows:

$$\nu_{r,q} = \mathbf{PV}_{2,q}\left(\pi, \gamma_2^\perp\left(\frac{r+q}{2}, 1\right)\right), \quad \overline{\nu}_{r,q} = \mathbf{PV}_{2,q}\left(\pi, \overline{\gamma_2^\perp}\left(\frac{r-q}{2}, -1\right)\right).$$

Two product bundles also enter in the description. For a $\mathbb{T}$-representation $V$, we let $\epsilon_q(V)$ denote the product bundle $pr_1: \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \times V \to \mathbf{PV}_{2,q}(\mathbb{C}^{n+1})$. Let $\mathbb{C}(s)$, for $s \in \mathbb{Z}$, denote the complex numbers $\mathbb{C}$ equipped with the $\mathbb{T}$-action $z * \lambda = z^s \lambda$, and equip the real numbers $\mathbb{R}$ with the trivial $\mathbb{T}$-action. The product bundles which enter are $\epsilon_q(\mathbb{R})$ and $\epsilon_q(\mathbb{C}(p))$. Note that $\epsilon_q(\mathbb{R})$ is a real $\mathbb{T}$ vector bundle and that the others are complex $\mathbb{T}$ vector bundles.

Write $\mathrm{Re}(z)$ for the real part of a complex number $z$. We have the following result, where the summands in Klingenberg's Theorem 11 have been labeled by an additional index $q$ indicating that they are vector bundles over $B_q(\mathbb{C}\mathrm{P}^n)$.

THEOREM 5.9. *Let $p$, $q$ and $r$ be positive integers with $p < q$ and $r < q$. There are isomorphisms of $\mathbb{T}$-vector bundles over the $\mathbb{T}$-equivariant diffeomorphism*

$$\phi_q\colon \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^n)$$

*as follows, where $h_q$ is defined for $q = 0 \bmod 2$ and $k_{r,q}$ is defined for $r = q \bmod 2$:*

$$\begin{aligned}
&f_q\colon \epsilon_q(\mathbb{R}) \to \eta_{h,0,q}; && f_q([u,v],s)(z) = sH(u,v)((\sqrt{z}\,)^q),\\
&g_q\colon \epsilon_q(\mathbb{C}(p)) \to \sigma_{h,p,q}; && g_q([u,v],\lambda)(z) = \mathrm{Re}(\lambda z^p)H(u,v)((\sqrt{z}\,)^q),\\
&h_q\colon \nu_{0,q} \to \eta_{v,0,q}; && h_q([u,v,w])(z) = V(u,v,w)((\sqrt{z}\,)^q),\\
&k_{r,q}\colon \nu_{r,s} \oplus \overline{\nu}_{r,s} \to \sigma_{v,r,q}; && k_{r,q}([u,v,w_1,w_2])(z)\\
& && \quad = V(u,v,(\sqrt{z}\,)^r w_1 + (\sqrt{z}\,)^{-r} w_2)((\sqrt{z}\,)^q).
\end{aligned}$$

*In the last formula, $\sqrt{z}$ appears three times. One must use the same choice of square root in each place.*

PROOF. For all four maps, the real dimension of the fiber of the domain equals the real dimension of the fiber of the codomain. So it suffices to show that each map is well-defined, surjective on fibers and $\mathbb{T}$-equivariant.

The map $f_q$ is independent of the choice of representative for the class $[u,v]$ and the choice of square root of $z$ by Proposition 5.4. So it is well-defined. By Lemma 3.2 and Lemma 5.1, $f_q$ is surjective on fibers. By Proposition 5.4 we see that it is $\mathbb{T}$-equivariant as follows:

$$\begin{aligned}
f_q([u,v],s)(z_1z_2) &= sH(u,v)((\sqrt{z_1})^q(\sqrt{z_2})^q)\\
&= sH(u,z_1^q v)((\sqrt{z_2})^q) = f_q(z_1 * [u,v],s)(z_2).
\end{aligned}$$

The map $g_q$ is well-defined by Proposition 5.4. For complex numbers $z_1 = \alpha_1 + i\beta_1$ and $z_2 = \alpha_2 + i\beta_2$ written in standard form, we have $\mathrm{Re}(z_1\overline{z}_2) = \alpha_1\alpha_2 + \beta_1\beta_2$. So for $\lambda = \alpha + i\beta$ and $z = e^{-2\pi i t}$, we get

$$\mathrm{Re}(\lambda z^p) = \alpha\cos(2\pi pt) + \beta\sin(2\pi pt)$$

so that $g_q$ is surjective on fibers by Lemma 3.2 and Lemma 5.1. We see that

$g_q$ is $\mathbb{T}$-equivariant as follows:

$$\begin{aligned} g_q([u,v],\lambda)(z_1z_2) &= \mathrm{Re}(\lambda(z_1z_2)^p) f_q([u,v],1)(z_1z_2) \\ &= \mathrm{Re}(z_1^p\lambda z_2^p) f_q(z_1 * [u,v],1)(z_2) \\ &= g_q(z_1 * ([u,v],\lambda))(z_2). \end{aligned}$$

The map $h_q$ is well-defined for $q$ even by Proposition 5.4. It is surjective on fibers by Lemma 3.2 and Lemma 5.1. By Proposition 5.4 we see that $h_q$ is $\mathbb{T}$-equivariant as follows:

$$\begin{aligned} h_q([u,v,w])(z_1z_2) &= V(u,v,w)((\sqrt{z_1})^q(\sqrt{z_2})^q) \\ &= V(u,z_1^q v, z_1^{\frac{q}{2}} w)((\sqrt{z_2})^q) = h_q(z_1 * [u,v,w])(z_2) \end{aligned}$$

Finally, consider the map $k_{r,q}$ where $r = q \bmod 2$. For $\lambda \in U(1)$ we have

$$[u,v,w_1,w_2] = [\lambda u, \lambda v, \lambda w_1, \lambda^{-1}\cdot w_2] = [\lambda u, \lambda v, \lambda w_1, \lambda w_2].$$

So by Proposition 5.4, the map $k_{r,q}$ is independent of the choice of representative of the class $[u,v,w_1,w_2]$. By the last remark of the proposition it is also independent of the choice of square root of $z$ and hence it is well-defined.

For $z = e^{2\pi it}$ with choice of square root $\sqrt{z} = e^{\pi it}$, we have

$$\begin{aligned} k_{r,q}([u,v,w_1,w_2])(z) &= V(u,v,e^{\pi rit}w_1 + e^{-\pi rit}w_2)(e^{\pi qit}) \\ &= \cos(r\pi t)V(u,v,w_1+w_2)(e^{\pi qit}) \\ &\qquad + \sin(r\pi t)V(u,v,i(w_1-w_2))(e^{\pi qit}). \end{aligned}$$

For a given pair of vectors $a$ and $b$ in $\{u,v\}^\perp$, the two equations $w_1 + w_2 = a$ and $i(w_1 - w_2) = b$ have the solution $w_1 = \frac{1}{2}(a - ib)$, $w_2 = \frac{1}{2}(a + ib)$. Comparing with Lemma 3.2 and Lemma 5.1, we see that the surjectivity on fibers holds.

Finally, we check that $k_{r,q}$ is $\mathbb{T}$-equivariant. Firstly, by Proposition 5.4 we have

$$\begin{aligned} &k_{r,q}([u,v,w_1,w_2])(z_1z_2) \\ &\quad = V\big(u,v,(\sqrt{z_1})^r(\sqrt{z_2})^r w_1 + (\sqrt{z_1})^{-r}(\sqrt{z_2})^{-r}w_2\big)\big((\sqrt{z_1})^q(\sqrt{z_2})^q\big) \\ &\quad = V\big(u,z_1^q v,(\sqrt{z_1})^{r+q}(\sqrt{z_2})^r w_1 + (\sqrt{z_1})^{-(r-q)}(\sqrt{z_2})^{-r}w_2\big)\big((\sqrt{z_2})^q\big). \end{aligned}$$

Secondly,

$$\begin{aligned} z_1 * [u,v,w_1,w_2] &= [u, z_1^q v, z_1^{(r+q)/2}w_1, z_1^{(r-q)/2}\cdot w_2] \\ &= [u, z_1^q v, (\sqrt{z_1})^{r+q}w_1, (\sqrt{z_1})^{-(r-q)}w_2], \end{aligned}$$

so that $k_{r,q}(z_1 * [u,v,w_1,w_2])(z_2)$ equals the above expression.

Combining Theorem 11 and Theorem 5.9 we obtain our first main result:

THEOREM 5.10. *For every positive integer $q$, there are isomorphisms of $\mathbb{T}$-vector bundles as follows:*

$$\mu_q^- \cong \begin{cases} \epsilon_q(\mathbb{R}) \oplus \bigoplus\limits_{0<s<q} \epsilon_q(\mathbb{C}(s)) \oplus \bigoplus\limits_{\substack{0<r<q \\ r=q \bmod 2}} (\nu_{r,q} \oplus \overline{\nu}_{r,q}), & \text{for } q \text{ odd,} \\ \epsilon_q(\mathbb{R}) \oplus \bigoplus\limits_{0<s<q} \epsilon_q(\mathbb{C}(s)) \oplus \nu_{0,q} \oplus \bigoplus\limits_{\substack{0<r<q \\ r=q \bmod 2}} (\nu_{r,q} \oplus \overline{\nu}_{r,q}), & \text{for } q \text{ even.} \end{cases}$$

## 6. Projective bundles and Borel constructions

In this section, we establish results which are aimed at calculating characteristic classes of the Borel construction with respect to the $\mathbb{T}$-action of the negative bundle.

PROPOSITION 6.1. *The following statements hold:*

(1) *Let $\xi_1 \to X$ and $\xi_2 \to X$ be $\mathbb{T}^2$-vector bundles and let $f\colon \mathbf{V}_{2,q}(\mathbb{C}^{n+1}) \to X$ be a $\mathbb{T}^2$-map. Then there is an isomorphism of $\mathbb{T}$-vector bundles*

$$\mathbf{PV}_{2,q}(f, \xi_1 \oplus \xi_2) \cong \mathbf{PV}_{2,q}(f, \xi_1) \oplus \mathbf{PV}_{2,q}(f, \xi_2).$$

(2) *Write $\epsilon_Y^k$ for the trivial $k$-dimensional complex vector bundle over a space $Y$. Equip $Y$ with the trivial $\mathbb{T}^2$-action. Let $g\colon \mathbf{V}_{2,q}(\mathbb{C}^{n+1}) \to Y$ be a $\mathbb{T}^2$-map and let $t\colon \mathbf{V}_{2,q}(\mathbb{C}^{n+1}) \to *$ denote the map to a point. Then for all integers $i$ and $j$ one has*

$$\mathbf{PV}_{2,q}(g, \epsilon_Y^k(i, j)) = \mathbf{PV}_{2,q}(t, \epsilon_*^k(i, j)) =: \mathbf{PV}_{2,q}(\epsilon^k(i, j)).$$

*Furthermore, there is a decomposition*

$$\mathbf{PV}_{2,q}(\epsilon^k(i, j)) = \bigoplus_{m=1}^{k} \mathbf{PV}_{2,q}(\epsilon^1(i, j)).$$

PROOF. (1) This is a special case of Proposition 5.5(3).

(2) Both pullbacks $f^*(\epsilon_Y^k(i, j))$ and $t^*(\epsilon_*^k(i, j))$ give the same $\mathbb{T}^2$-vector bundle. Its projection map is $pr_1\colon \mathbf{V}_{2,q}(\mathbb{C}^{n+1}) \times \mathbb{C}^k \to \mathbf{V}_{2,q}(\mathbb{C}^{n+1})$ and the action on the total space is given by $(z_1, z_2)*((u, v), w) = ((z_2 u, z_1^q z_2 v), z_1^i z_2^j w)$. This observation gives us the first part of the statement. The last part follows from (1).

Let $\gamma_1 \to \mathbb{C}\mathrm{P}^n$ be the canonical line bundle. Its total space consists of pairs $(V, v)$, where $V$ is a complex one-dimensional subspace of $\mathbb{C}^{n+1}$ and $v \in V$. The projection map sends $(V, v)$ to $V \in \mathbb{C}\mathrm{P}^n$. Sometimes we use the quotient space model $S^{n+1}/U(1)$ for $\mathbb{C}\mathrm{P}^n$ instead. Then a point in $\mathbb{C}\mathrm{P}^n$ is written as $[u]$, where $u \in S^{2n+1}$.

DEFINITION 6.2. Let $\pi_i$ for $i = 1, 2$ be the compositions

$$\pi_i \colon \mathbf{V}_{2,q}(\mathbb{C}^{n+1}) \longrightarrow \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \xrightarrow{pr_i} \mathbb{C}\mathrm{P}^n,$$

where $pr_1([u, v]) = [u]$ and $pr_2([u, v]) = [v]$. Note that $\pi_i$ is a $\mathbb{T}^2$-map, where the action on $\mathbb{C}\mathrm{P}^n$ is trivial. For $r = q \bmod 2$, we define one dimensional complex $\mathbb{T}$-vector bundles over $\mathbf{PV}_{2,q}(\mathbb{C}^{n+1})$ by

$$L_{0,r,q} = \mathbf{PV}_{2,q}\left(\epsilon^1\left(\frac{r+q}{2}, 1\right)\right), \qquad \overline{L}_{0,r,q} = \mathbf{PV}_{2,q}\left(\overline{\epsilon}^1\left(\tfrac{r-q}{2}, -1\right)\right),$$

$$L_{1,r,q} = \mathbf{PV}_{2,q}\left(\pi_1, \gamma_1\left(\frac{r+q}{2}, 1\right)\right), \quad \overline{L}_{1,r,q} = \mathbf{PV}_{2,q}\left(\pi_1, \overline{\gamma}_1\left(\tfrac{r-q}{2}, -1\right)\right),$$

$$L_{2,r,q} = \mathbf{PV}_{2,q}\left(\pi_2, \gamma_1\left(\frac{r+q}{2}, 1\right)\right), \quad \overline{L}_{2,r,q} = \mathbf{PV}_{2,q}\left(\pi_2, \overline{\gamma}_1\left(\tfrac{r-q}{2}, -1\right)\right).$$

THEOREM 6.3. *There are $\mathbb{T}$-equivariant isomorphisms for $r = q \bmod 2$ as follows:*

$$\nu_{r,q} \oplus L_{1,r,q} \oplus L_{2,r,q} \cong L_{0,r,q}^{\oplus(n+1)},$$

$$\overline{\nu}_{r,q} \oplus \overline{L}_{1,r,q} \oplus \overline{L}_{2,r,q} \cong \overline{L}_{0,r,q}^{\oplus(n+1)}.$$

PROOF. We give the proof of the first isomorphism. The proof of the second is similar. Put $m = (r + q)/2$. By Proposition 6.1, we have

$$\begin{aligned}\mathbf{PV}_{2,q}(\pi, \gamma_2^{\perp}(m, 1)) \oplus \mathbf{PV}_{2,q}(\pi, \gamma_2(m, 1)) &\cong \mathbf{PV}_2(\pi, \gamma_2^{\perp}(m, 1) \oplus \gamma_2(m, 1)) \\ &\cong \mathbf{PV}_2(\pi, \epsilon^{n+1}(m, 1)) \cong L_{0,r,q}^{\oplus(n+1)}.\end{aligned}$$

Thus it suffices to show that $\mathbf{PV}_{2,q}(\pi, \gamma_2(m, 1)) \cong L_{1,r,q} \oplus L_{2,r,q}$. The standard isomorphism

$$\pi_1^*(\gamma_1) \oplus \pi_2^*(\gamma_1) \to \pi^*(\gamma_2); \quad \big((u, v, w_1), (u, v, w_2)\big) \to (u, v, w_1 + w_2),$$

where $w_1 \in \mathrm{span}_{\mathbb{C}}(u)$ and $w_2 \in \mathrm{span}_{\mathbb{C}}(v)$, is $\mathbb{T}^2$-equivariant with respect to our actions. The result follows by Proposition 5.5(2).

We will now give pullback descriptions of the $\mathbb{T}$-line bundles. The following notation is used: for a complex vector bundle $\xi \to X$ and integer $m \in \mathbb{Z}$, we put $\xi(m) = \xi$, where $z \in \mathbb{T} \subseteq \mathbb{C}$ acts on each fiber by multiplication with $z^m$. Thus, $\xi(m) \to X$ is a $\mathbb{T}$-vector bundle over a trivial $\mathbb{T}$-space.

PROPOSITION 6.4. *Let $\epsilon^1 \to \mathbb{C}\mathrm{P}^n$ be the trivial line bundle and $\gamma_1 \to \mathbb{C}\mathrm{P}^n$ the canonical line bundle. There are pullback diagrams of $\mathbb{T}$-vector bundles as follows for $r = q \bmod 2$ and $i = 1, 2$:*

$$\begin{array}{ccc} L_{i,r,q} & \longrightarrow & \epsilon^1\left(\frac{r+(-1)^{i+1}q}{2}\right) \\ \downarrow & & \downarrow \\ \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) & \xrightarrow{pr_i} & \mathbb{C}\mathrm{P}^n, \end{array} \qquad \begin{array}{ccc} L_{0,r,q} & \longrightarrow & \overline{\gamma}_1\left(\frac{r+(-1)^{i+1}q}{2}\right) \\ \downarrow & & \downarrow \\ \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) & \xrightarrow{pr_i} & \mathbb{C}\mathrm{P}^n, \end{array}$$

$$\begin{array}{ccc} \overline{L}_{i,r,q} & \longrightarrow & \overline{\epsilon}^1\left(\frac{r+(-1)^{i}q}{2}\right) \\ \downarrow & & \downarrow \\ \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) & \xrightarrow{pr_i} & \mathbb{C}\mathrm{P}^n, \end{array} \qquad \begin{array}{ccc} \overline{L}_{0,r,q} & \longrightarrow & \gamma_1\left(\frac{r+(-1)^{i}q}{2}\right) \\ \downarrow & & \downarrow \\ \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) & \xrightarrow{pr_i} & \mathbb{C}\mathrm{P}^n. \end{array}$$

It might seems strange that e.g. the bundle $L_{1,r,q}$ is a trivial $\mathbb{T}$-vector bundle over $\mathbf{PV}_{2,q}(\mathbb{C}^{n+1})$ as stated. It did not come from a trivial bundle but from $\gamma_1$. The 'untwisting' appears when the quotient is formed in the construction of $L_{1,r,q}$ as a result of the definition of the $U(1)$-action.

PROOF. Regarding the upper-left pullback diagram for $i = 1$, the bundle map over $pr_1$ is defined by

$$f_1 \colon L_{1,r,q} \to \mathbb{C}\mathrm{P}^n \times \mathbb{C}; \quad [u, v, w] \mapsto ([u], k(w, u)),$$

where $k(w, u) \in \mathbb{C}$ is the scalar determined by $w = k(w, u)u$. The following properties hold for $w_1, w_2 \in \mathrm{span}_{\mathbb{C}}(u)$, $a_1, a_2 \in \mathbb{C}$ and $b \in U(1)$:

$$k(a_1 w_1 + a_2 w_2, u) = a_1 k(w_1, u) + a_2 k(w_2, u),$$
$$k(w, bu) = b^{-1} k(w, u).$$

It follows that $k(zw, zu) = k(w, u)$, for $z \in U(1)$, so the bundle map $f_1$ is well-defined:

$$[zu, zv, zw] \mapsto ([zu], k(zw, zu)) = ([u], k(w, u)).$$

Furthermore, $f_1$ is a fiber-wise $\mathbb{C}$-linear isomorphism, so we have a pullback. We check that $f_1$ is $\mathbb{T}$-equivariant as well: put $m = (r + q)/2$. Then,

$$f_1(z*[u, v, w]) = f_1([u, z^q v, z^m w]) = ([u], k(z^m w, u)) = ([u], z^m k(w, u)).$$

Similarly, the bundle map $f_2\colon L_{2,r,q} \to \mathbb{C}\mathrm{P}^n \times \mathbb{C}$; $[u, v, w] \mapsto ([v], k(w, v))$, where $w = k(w, v)v$, gives us the upper-left pullback diagram for $i = 2$. In this case, the $\mathbb{T}$-equivariance follows from the computation

$$\begin{aligned} f_2(z * [u, v, w]) &= f_2([u, z^q v, z^m w]) = ([z^q v], k(z^m w, z^q v)) \\ &= ([v], z^{m-q} k(w, u)). \end{aligned}$$

The bundle maps in the lower-left diagram are still $f_1$ and $f_2$, but with conjugate complex structure on domain and target. For $i = 1$, we have

$$\begin{aligned} f_1(z * [u, v, w]) &= f_1([u, z^q v, z^{m-q} \cdot w]) = f_1([u, z^q v, z^{-m+q} w]) \\ &= ([u], k(z^{-m+q} w, u)) = ([u], z^{-m+q} k(w, u)) \\ &= ([u], z^{m-q} \cdot k(w, u)). \end{aligned}$$

Thus, $f_1$ is $\mathbb{T}$-equivariant. A similar argument gives us that $f_2$ is $\mathbb{T}$-equivariant, so we have the stated pullback diagrams for $i = 1, 2$.

The bundle map in the upper-right diagram for $i = 1$ is defined by

$$g_1\colon L_0 \to \overline{\gamma}_1; \quad [u, v, k] \mapsto ([u], k \cdot u) = ([u], \overline{k}u).$$

It is well-defined because $z\overline{z} = 1$, for $z \in U(1)$, implies that

$$[zu, zv, zk] \mapsto ([zu], \overline{z}\overline{k}zu) = ([u], \overline{k}u).$$

Since $g_1$ is a fiber-wise isomorphism, we have a pullback. $g_1$ is also $\mathbb{T}$-equivariant:

$$g_1(z * [u, v, k]) = g_1([u, z^q v, z^m k]) = ([u], z^{-m}\overline{k}u) = ([u], z^m \cdot \overline{k}u).$$

Similarly, the bundle map $g_2\colon L_0 \to \overline{\gamma}_1$; $[u, v, k] \mapsto ([v], \overline{k}v)$ gives us the upper-right pullback diagram for $i = 2$.

The bundle maps $g_1$ and $g_2$ with conjugate complex structure on domain and target, gives the lower-right pullback diagrams.

We are interested in the vector bundle $E\mathbb{T} \times_{\mathbb{T}} \mu_q^-$. Fortunately, forming Borel constructions of $G$-vector bundles is well-behaved with respect to Whitney sums and pullbacks. One has the following standard results:

Proposition 6.5. *Let $G$ be a compact Lie group and let $\xi$, $\eta$ be $G$-vector bundles over a $G$-space $X$. Then there is a natural isomorphism*

$$EG \times_G (\xi \oplus \eta) \xrightarrow{\cong} (EG \times_G \xi) \oplus (EG \times_G \eta).$$

*Furthermore, if $f\colon Y \to X$ is a $G$-map, then there is a natural isomorphism*

$$EG \times_G f^*(\xi) \xrightarrow{\cong} (EG \times_G f)^*(EG \times_G \xi).$$

Corollary 6.6. *Let $G$ be a compact Lie group and $p\colon \xi \to X$ a $G$-vector bundle over a trivial $G$-space $X$. Write $EG \to BG$ for the universal principal $G$-bundle, and let $i_1\colon BG \to BG \times X$ be the inclusion $b \mapsto (b, x_0)$ where $x_0 \in X$. Then there is a pullback diagram*

$$\begin{tikzcd} EG \times_G p^{-1}(x_0) \arrow[r] \arrow[d, "pr_i"] & EG \times_G \xi \arrow[d, "EG\times_G p"] \\ BG \arrow[r, "i_1"] & BG \times X. \end{tikzcd}$$

## 7. Characteristic classes

In this section we compute the Chern classes of the vector bundles $(\mu_q^-)_{h\mathbb{T}}$. By Theorem 6.3 and Proposition 6.4, the following result is relevant:

Proposition 7.1. *Let $x = c_1(\gamma_1)$ and $u = c_1(\gamma_1^\infty)$ be the first Chern classes of the canonical line bundles $\gamma_1 \to \mathbb{C}\mathrm{P}^n$ and $\gamma_1^\infty \to \mathbb{C}\mathrm{P}^\infty = B\mathbb{T}$, so that*

$$H^*(B\mathbb{T} \times \mathbb{C}\mathrm{P}^n; \mathbb{Z}) = \mathbb{Z}[u] \otimes \mathbb{Z}[x]/(x^{n+1}).$$

*Let $\epsilon^1 \to \mathbb{C}\mathrm{P}^n$ be the trivial line bundle. Then for every $m \in \mathbb{Z}$ we have*

$$c_1(E\mathbb{T} \times_{\mathbb{T}} \gamma_1(m)) = mu \otimes 1 + 1 \otimes x,$$
$$c_1(E\mathbb{T} \times_{\mathbb{T}} \epsilon^1(m)) = mu \otimes 1.$$

Proof. We start by proving the following claim:

$$c_1(E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}(m)) = mu.$$

The first Chern class defines a group homomorphism

$$c_1\colon (\mathrm{Vect}^1_{\mathbb{C}}(B\mathbb{T}), \otimes, \overline{(\ )}) \to (H^2(B\mathbb{T}; \mathbb{Z}), +, -),$$

which is in fact an isomorphism since $B\mathbb{T}$ is homotopy equivalent to the CW-complex $\mathbb{C}\mathrm{P}^\infty$ (see [8, page 250] or [6]). There are isomorphisms of vector bundles, for every $n$, as follows:

$$S^{2n-1} \times_{\mathbb{T}} \mathbb{C}(1) \to \gamma_1; \quad [v, z] \mapsto (\mathrm{span}_{\mathbb{C}}(v), zv),$$
$$S^{2n-1} \times_{\mathbb{T}} \mathbb{C}(-1) \to \overline{\gamma}_1; \quad [v, z] \mapsto (\mathrm{span}_{\mathbb{C}}(v), \overline{z}v).$$

Thus, we have isomorphisms $E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}(1) \cong \gamma_1$ and $E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}(-1) \cong \overline{\gamma}_1$. Note that $\mathbb{C}(0)$ equals $\mathbb{C}$ with trivial $\mathbb{T}$-action and, for $k > 0$, we have that $\mathbb{C}(k) \cong \otimes_{i=1}^k \mathbb{C}(1)$ and $\mathbb{C}(-k) \cong \otimes_{i=1}^k \mathbb{C}(-1)$. We get corresponding tensor product decompositions of the vector bundles $E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}(m)$. The claim follows.

Choose base points in $B\mathbb{T}$ and $\mathbb{C}\mathrm{P}^n$, and consider the associated inclusions

$$i_1 \colon B\mathbb{T} \to B\mathbb{T} \times \mathbb{C}\mathrm{P}^n, \quad i_2 \colon \mathbb{C}\mathrm{P}^n \to B\mathbb{T} \times \mathbb{C}\mathrm{P}^n.$$

By Corollary 6.6, the pullbacks of both $\gamma_1(m)_{h\mathbb{T}}$ and $\epsilon^1(m)_{h\mathbb{T}}$ along $i_1$ equal the line bundle $E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}(m)$. Thus,

$$i_1^*(c_1(\gamma_1(m)_{h\mathbb{T}})) = i_1^*(c_1((\epsilon^1(m))_{h\mathbb{T}})) = c_1(E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}(m)) = mu.$$

The pullback of $\gamma_1(m)_{h\mathbb{T}}$ along $i_2 \colon \mathbb{C}\mathrm{P}^n \to E\mathbb{T} \times \mathbb{C}\mathrm{P}^n \to B\mathbb{T} \times \mathbb{C}\mathrm{P}^n$ equals $\gamma_1$ and the pullback of $\epsilon^1(m)_{h\mathbb{T}}$ along $i_2$ is the trivial line bundle $\epsilon^1$. Thus,

$$i_2^*(c_1(\gamma_1(m)_{h\mathbb{T}})) = x, \quad i_2^*(c_1(\epsilon^1(m)_{h\mathbb{T}})) = 0.$$

Finally, $H^2(B\mathbb{T} \times \mathbb{C}\mathrm{P}^n; \mathbb{Z})$ is generated by the two classes $u \otimes 1$, $1 \otimes x$ and

$$\begin{aligned} i_1^*(u \otimes 1) &= i_1^* \circ pr_1^*(u) = u, \quad & i_2^*(u \otimes 1) &= i_2^* \circ pr_1^*(u) = 0, \\ i_1^*(1 \otimes x) &= i_1^* \circ pr_2^*(x) = 0, \quad & i_2^*(1 \otimes x) &= i_2^* \circ pr_2^*(x) = x, \end{aligned}$$

so we have the desired result.

Remark 7.2. For any complex vector bundle $\xi$ one has that

$$\overline{E\mathbb{T} \times_{\mathbb{T}} \xi(m)} = E\mathbb{T} \times_{\mathbb{T}} \overline{\xi}(-m),$$

since in both cases, we mod out by the equivalence relation $(ez, v) \sim (e, z^m v)$, and we have the conjugate complex structure. So by the above result

$$\begin{aligned} c_1(E\mathbb{T} \times_{\mathbb{T}} \overline{\gamma}_1(m)) &= mu \otimes 1 - 1 \otimes x, \\ c_1(E\mathbb{T} \times_{\mathbb{T}} \overline{\epsilon}^1(m)) &= mu \otimes 1. \end{aligned}$$

In order to use the pullback diagrams of Proposition 6.4, we must compute the induced maps in cohomology of the two projection maps

$$(pr_i)_{h\mathbb{T}} \colon \mathbf{PV}_{2,q}(\mathbb{C}^{n+1})_{h\mathbb{T}} \to (\mathbb{C}\mathrm{P}^n)_{h\mathbb{T}} = B\mathbb{T} \times \mathbb{C}\mathrm{P}^n, \quad i = 1, 2.$$

The mod $p$ cohomology of the domain space was computed in [15]. We will need some of the results leading to this calculation.

Let $\pi \colon \mathbb{P}(\gamma_2) \to \mathbf{G}_2(\mathbb{C}^{n+1})$ denote the projective bundle of the canonical bundle $\gamma_2 \to \mathbf{G}_2(\mathbb{C}^{n+1})$. We can describe the total space as a set of flags:

$$\mathbb{P}(\gamma_2) = \{ V_1 \subseteq V_2 \subseteq \mathbb{C}^{n+1} \mid \dim_{\mathbb{C}}(V_i) = i \}.$$

By [15] Lemma 2.6, we have an isomorphism

$$\begin{aligned}\psi\colon \mathbf{PV}_{2,1}(\mathbb{C}^{n+1})/\mathbb{T} &\xrightarrow{\cong} \mathbb{P}(\gamma_2);\\ [u,v]\mathbb{T} &\longmapsto \bigl(\operatorname{span}_{\mathbb{C}}(u) \subseteq \operatorname{span}_{\mathbb{C}}(u,v) \subseteq \mathbb{C}^{n+1}\bigr).\end{aligned}$$

There is a canonical line bundle $\lambda \to \mathbb{P}(\gamma_2)$ with complementary line bundle $\lambda' \to \mathbb{P}(\gamma_2)$ as follows:

$$\lambda = \{\, (V_1 \subseteq V_2, v) \mid v \in V_1 \,\}, \quad \lambda' = \{\, (V_1 \subseteq V_2, w) \mid w \in V_1^{\perp} \subseteq V_2 \,\}.$$

There are pullback diagrams

$$\begin{tikzcd} \lambda \arrow[r] \arrow[d] & \gamma_1 \arrow[d] \\ \mathbb{P}(\gamma_2) \arrow[r, "p_1"] & \mathbb{C}\mathrm{P}^n, \end{tikzcd} \qquad \begin{tikzcd} \lambda' \arrow[r] \arrow[d] & \gamma_1 \arrow[d] \\ \mathbb{P}(\gamma_2) \arrow[r, "p_2"] & \mathbb{C}\mathrm{P}^n, \end{tikzcd}$$

where $p_1(V_1 \subseteq V_2) = V_1$ and $p_2(V_1 \subseteq V_2) = V_1^{\perp}$. Note also that $\lambda \oplus \lambda' \cong \pi^*(\gamma_2)$. We have the following slightly enhanced version of Theorem 3.2 in [15]:

THEOREM 7.3. *There is an isomorphism of graded rings*

$$H^*(\mathbb{P}(\gamma_2); \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2]/(Q_n, Q_{n+1}),$$

*where $x_1$ and $x_2$ have degree 2 and for positive integers $k$,*

$$Q_k(x_1, x_2) = \sum_{i=1}^{k} x_1^i x_2^{k-i} = \frac{x_1^{k+1} - x_2^{k+1}}{x_1 - x_2}.$$

*Furthermore, $p_1^*(x) = x_1$ and $p_2^*(x) = x_2$.*

PROOF. The ring structure is given in Theorem 3.2 of [15]. From the proof of this theorem one has that

$$x_1 = c_1(\lambda), \quad x_2 = \pi^*(c_1(\gamma_2)) - c_1(\lambda), \quad \pi^*(c_1(\gamma_2)) = x_1 + x_2.$$

Thus, $p_1^*(x) = p_1^*(c_1(\gamma_1)) = c_1(p_1^*(\gamma_1)) = c_1(\lambda) = x_1$ and

$$x_1 + x_2 = c_1(\pi^*(\gamma_2)) = c_1(\lambda \oplus \lambda') = c_1(\lambda) + c_1(\lambda') = x_1 + c_1(\lambda'),$$

so that $x_2 = c_1(\lambda')$.

Recall that a left $G$-space $X$ is also a right $G$-space with action $x * g = g^{-1} * x$ for $x \in X$, $g \in G$. For the right $\mathbb{T}$-space $\mathbf{PV}_{2,1}(\mathbb{C}^{n+1})$ we have the following result:

LEMMA 7.4. *The principal $\mathbb{T}$-bundle $\rho\colon \mathbf{PV}_{2,1}(\mathbb{C}^{n+1}) \to \mathbf{PV}_{2,1}(\mathbb{C}^{n+1})/\mathbb{T}$ has associated complex line bundle $\lambda \otimes_{\mathbb{C}} \overline{\lambda'}$. That is, we have an isomorphism of line bundles*

$$\begin{tikzcd} \mathbf{PV}_{2,1}(\mathbb{C}^{n+1}) \times_{\mathbb{T}} \mathbb{C}(1) \arrow[r, "\cong"] \arrow[d] & \lambda \otimes_{\mathbb{C}} \overline{\lambda'} \arrow[d] \\ \mathbf{PV}_{2,1}(\mathbb{C}^{n+1})/\mathbb{T} \arrow[r, "\psi", "\cong"'] & \mathbb{P}(\gamma_2), \end{tikzcd}$$

*The Euler class of $\rho$ is*

$$e(\rho) = x_1 - x_2.$$

PROOF. The bundle map over the isomorphism $\psi$ is defined by

$$\big[[u, v], k\big] \mapsto \big((\mathrm{span}_{\mathbb{C}}(u) \subseteq \mathrm{span}_{\mathbb{C}}(u, v)), k(u \otimes v)\big).$$

We check that this is a well-defined map. Firstly, the linear span is unchanged by a rescaling of the generators by non-zero scalars. Secondly, for $z \in U(1)$ we have $[u, v] = [zu, zv]$ in the projective Stiefel manifold, but also

$$zu \otimes zv = zu \otimes \overline{z} \cdot v = z\overline{z}(u \otimes v) = u \otimes v.$$

Thirdly, for $z \in \mathbb{T}$ we have $[[u, v], zk] = [[u, v] * z, k] = [[zu, v], k]$ but also $zk(u \otimes v) = k(zu \otimes v)$ which completes the argument. The bundle map is an isomorphism on fibers.

The Euler class of $\rho$ equals the first Chern class of the associated line bundle, which is $c_1(\lambda \otimes_{\mathbb{C}} \overline{\lambda'}) = c_1(\lambda) - c_1(\lambda') = x_1 - x_2$.

REMARK 7.5. By the lemma above we get a sphere bundle interpretation of the projective Stiefel manifold:

$$\mathbf{PV}_{2,1}(\mathbb{C}^{n+1}) = \mathbf{PV}_{2,1}(\mathbb{C}^{n+1}) \times_{\mathbb{T}} \mathbb{T} = S(\mathbf{PV}_{2,1}(\mathbb{C}^{n+1}) \times_{\mathbb{T}} \mathbb{C}(1)) \cong S(\lambda \otimes_{\mathbb{C}} \overline{\lambda'}).$$

Thus, there is an isomorphism of left $\mathbb{T}$-spaces for every $q \in \mathbb{Z}$:

$$\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \cong S\big((\lambda \otimes_{\mathbb{C}} \overline{\lambda'})(-q)\big).$$

For a left $\mathbb{T}$-space $X$ with action map $\mu\colon \mathbb{T} \times X \to X$, we can twist the action by the power map $\lambda_n\colon \mathbb{T} \to \mathbb{T}$; $\lambda_n(z) = z^n$ and obtain another $\mathbb{T}$-space $X^{(n)}$. Thus the underlying spaces of $X$ and $X^{(n)}$ are equal, but the action map for $X^{(n)}$ is $\mu_n\colon \mathbb{T} \times X^{(n)} \to X^{(n)}$; $\mu_n(z, x) = \mu(\lambda_n(x), z)$.

Proposition 7.6. *Let $X$ be a left $\mathbb{T}$-space and let $C_n$ denote the cyclic group of order $n$. There is a vertical and horizontal pullback of fibration sequences which is natural in $X$ as follows:*

$$\begin{tikzcd}
\star \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d] & X \arrow[d] \\
BC_n \arrow[r] \arrow[d, equal] & E\mathbb{T} \times_{\mathbb{T}} X^{(n)} \arrow[r, "E(\lambda_n)\times_{\mathbb{T}}\mathrm{id}"] \arrow[d, "pr_1"] & E\mathbb{T} \times_{\mathbb{T}} X \arrow[d, "pr_1"] \\
BC_n \arrow[r] & B\mathbb{T} \arrow[r, "B(\lambda_n)"] & B\mathbb{T}
\end{tikzcd}$$

*Assume furthermore that the right $\mathbb{T}$-space associated to $X$ gives a principal $\mathbb{T}$-bundle $\rho\colon X \to X/\mathbb{T}$. Write it as a pullback of the universal bundle $E\mathbb{T} \to B\mathbb{T}$ along a map $f\colon X/\mathbb{T} \to B\mathbb{T}$. Then the right vertical projection map in the diagram above can be replaced by $f$ in the following sense: there is a diagram, which commutes up to homotopy, and where $pr_2$ is a weak homotopy equivalence*

$$\begin{tikzcd}
E\mathbb{T} \times_{\mathbb{T}} X \arrow[r, "pr_2", "\simeq"'] \arrow[d, "pr_1"] & X/\mathbb{T} \arrow[dl, "f"] \\
B\mathbb{T} &
\end{tikzcd}$$

*Finally, if we let $e(\rho)$ denote the Euler class, the two maps*

$$H^*(B\mathbb{T};\mathbb{Z}) \xrightarrow{pr_1^*} H^*(E\mathbb{T} \times_{\mathbb{T}} X^{(n)};\mathbb{Z}) \xleftarrow{pr_2^*} H^*(X/\mathbb{T};\mathbb{Z})$$

*satisfy*

$$pr_1^*(nu) = pr_2^*(e(\rho)).$$

Proof. A proof for the first pullback diagram can be found in [15] Lemma 6.1. Regarding the second diagram, first note that $pr_2$ is a fibration with contractible fiber $E\mathbb{T}$ and hence a weak homotopy equivalence. In order to verify that the diagram commutes up to homotopy, it suffices to check, that the right triangle in the following diagram commutes up to homotopy:

$$\begin{tikzcd}
E\mathbb{T} \times_{\mathbb{T}} X \arrow[r, "E\mathbb{T}\times_{\mathbb{T}}\hat{f}"] \arrow[d, "pr_2"] & E\mathbb{T} \times_{\mathbb{T}} E\mathbb{T} \arrow[r, "pr_1"] \arrow[d, "pr_2"] & B\mathbb{T} \arrow[dl, equal] \\
X/\mathbb{T} \arrow[r, "f"] & B\mathbb{T} &
\end{tikzcd}$$

Both $pr_1$ and $pr_2$ in the triangle are homotopy equivalences. By the diagrams

$$\begin{tikzcd} E\mathbb{T}\times_{\mathbb{T}} E\mathbb{T} \arrow[r, "pr_1"] \arrow[d, "tw"] & B\mathbb{T} \arrow[d, equal] \\ E\mathbb{T}\times_{\mathbb{T}} E\mathbb{T} \arrow[r, "pr_1"] & B\mathbb{T} \end{tikzcd} \qquad \begin{tikzcd} {[E\mathbb{T}\times_{\mathbb{T}} E\mathbb{T}, B\mathbb{T}]} \arrow[r, "\cong"] \arrow[d, "tw^*"] & H^2(E\mathbb{T}\times_{\mathbb{T}} E\mathbb{T};\mathbb{Z}) = \mathbb{Z} \arrow[d, "tw^*"] \\ {[E\mathbb{T}\times_{\mathbb{T}} E\mathbb{T}, B\mathbb{T}]} \arrow[r, "\cong"] & H^2(E\mathbb{T}\times_{\mathbb{T}} E\mathbb{T};\mathbb{Z}) = \mathbb{Z} \end{tikzcd}$$

it suffices to see that $tw^* = \mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$. The twist gives a self map of the fibration

$$\mathbb{T} \longrightarrow E\mathbb{T}\times E\mathbb{T} \longrightarrow E\mathbb{T}\times_{\mathbb{T}} E\mathbb{T}.$$

It is the identity on the fiber of the point $[a, a]$ since the twist also changes the sides of the actions on both factors. By the long exact sequence of homotopy groups, one sees that $tw_* = \mathrm{id}$ on $\pi_2(E\mathbb{T}\times_{\mathbb{T}} E\mathbb{T})$. By Hurewicz and universal coefficients, the result follows for cohomology.

We have that $f^*(u) = e(\rho)$. In the second diagram of the theorem, this gives us that $pr_1^*(u) = pr_2^*(e(\rho))$. Combining this with the first diagram, the last statement follows.

PROPOSITION 7.7. *There is a commutative diagram, for $i = 1, 2$, where $\pi_1$ and $\pi_2$ denotes projection on first and second factor:*

$$\begin{tikzcd} \mathbb{P}(\gamma_2) \arrow[r, "p_i"] & \mathbb{C}P^n \\ \mathbf{PV}_{2,1}(\mathbb{C}^{n+1})/\mathbb{T} \arrow[u, "\cong"] \arrow[r, "E\mathbb{T}\times_{\mathbb{T}} pr_i/\mathbb{T}"] & \mathbb{C}P^n \arrow[u, equal] \\ E\mathbb{T}\times_{\mathbb{T}} \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \arrow[u, "\pi_2"] \arrow[r, "E\mathbb{T}\times_{\mathbb{T}} pr_i"] \arrow[d, "\pi_1"] & B\mathbb{T}\times\mathbb{C}P^n \arrow[u, "\pi_2"] \arrow[d, "\pi_1"] \\ B\mathbb{T} \arrow[r, equal] & B\mathbb{T} \end{tikzcd}$$

*In cohomology with $\mathbb{Z}$-coefficients, one has that*

$$(E\mathbb{T}\times_{\mathbb{T}} pr_i)^*(1\otimes x) = \pi_2^*(x_i) \quad \textit{and} \quad (E\mathbb{T}\times_{\mathbb{T}} pr_i)^*(qu\otimes 1) = \pi_2^*(x_1 - x_2).$$

PROOF. Only the top square in the diagram requires an argument and it commutes by direct verification. The first equation follows by the diagram. The second follows by Lemma 7.4 and Proposition 7.6.

We can now prove the following enhanced version of [15] Theorem 4.1:

THEOREM 7.8. *Let $n$ and $q$ be integers with $n > 1$ and $q > 0$. Let $p$ be a prime. There is an isomorphism*

$$H^*_{\mathbb{T}}(\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}); \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p[x_1, x_2]/(Q_n, Q_{n+1}), & p \nmid q, \\ \mathbb{F}_p[u, x, \sigma]/(x^{n+1}, \sigma^2), & p \mid q,\ p \mid (n+1), \\ \mathbb{F}_p[u, x, \overline{\sigma}]/(x^n, \overline{\sigma}^2), & p \mid q,\ p \nmid (n+1), \end{cases}$$

*where the classes $u$, $x$, $x_1$, $x_2$ have degree* 2 *and* $\deg(\sigma) = 2n - 1$, $\deg(\overline{\sigma}) = 2n + 1$. *The polynomials $Q_k \in \mathbb{F}_p[x_1, x_2]$ are defined as follows for positive integers $k$:*

$$Q_k(x_1, x_2) = \sum_{i=0}^{k} x_1^i x_2^{k-i}.$$

*The maps*

$$pr_i^*\colon H^*(B\mathbb{T} \times \mathbb{C}\mathrm{P}^n; \mathbb{F}_p) \longrightarrow H^*_{\mathbb{T}}(\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}); \mathbb{F}_p)$$

*are given by the following for $i = 1, 2$:*

$$\begin{aligned} &u \otimes 1 \mapsto \frac{1}{q}(x_1 - x_2), \quad && 1 \otimes x \mapsto x_i, \quad && \text{for } p \nmid q, \\ &u \otimes 1 \mapsto u, && 1 \otimes x \mapsto x, && \text{for } p \mid q. \end{aligned}$$

PROOF. The computation of the cohomology ring is given in [15] Theorem 4.1. We review parts of the proof in order to include the description of the projection maps.

By Proposition 7.6, we have a pullback of fibration sequences

$$\begin{tikzcd} BC_q \arrow[r] \arrow[d, equal] & E\mathbb{T} \times_{\mathbb{T}} \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbf{PV}_{2,1}(\mathbb{C}^{n+1})/\mathbb{T} \arrow[d, "f"] \\ BC_q \arrow[r] & B\mathbb{T} \arrow[r, "B(\lambda_q)"] & B\mathbb{T} \end{tikzcd}$$

Consider the associated Serre spectral sequences. We have trivial coefficients in both of these since the base of the lower fibration is simply connected.

Assume that $p \nmid q$. Then, $H^*(BC_q; \mathbb{F}_p) = \mathbb{F}_p$, and by the upper spectral sequence $\pi_2$ induces an isomorphism in cohomology. The results follows by Theorem 7.3 and Proposition 7.7 via universal coefficients.

Assume that $p \mid q$. One has that $H^*(BC_q; \mathbb{F}_p) = \mathbb{F}_p[v, w]/I_{p,q}$, where the degrees are $|v| = 1$, $|w| = 2$ and $I_{p,q}$ is the ideal $(v^2 - w)$ for $p = 2$, $4 \nmid q$

and the ideal $(v^2)$ otherwise. The $E_2$-page of the Serre spectral sequence for the upper fibration has the form

$$E_2^{**} = \mathbb{F}_p[x_1, x_2]/(Q_n, Q_{n+1}) \otimes \mathbb{F}_p[v, w]/I_{p,q},$$

where the bi-degrees are $\|x_1\| = \|x_2\| = (2, 0)$, $\|v\| = (0, 1)$, $\|w\| = (0, 2)$. Via the spectral sequence for the lower fibration sequence, one finds that $d_2(w) = 0$, $d_2(v) = x_1 - x_2$ and that $w$ is a permanent cycle. It follows that $E_3 = E_\infty$.

We let $K_n$ and $C_n$ denote the kernel and cokernel of multiplication with $(x_1 - x_2)$ on $\mathbb{F}_p[x_1, x_2]/(Q_n, Q_{n+1})$. Then

$$E_\infty^{**} = E_3^{**} = (C_n \oplus vK_n) \otimes \mathbb{F}_p[w].$$

In [15], proof of Theorem 4.1, the kernel and cokernel are analyzed further, and one obtains the following bigraded algebra description of the $E_\infty$-page:

$$E_\infty^{**} = \begin{cases} \mathbb{F}_p[w, x_1, \sigma]/(x_1^{n+1}, \sigma^2), & p \mid (n+1), \\ \mathbb{F}_p[w, x_1, \overline{\sigma}]/(x_1^n, \overline{\sigma}^2), & p \nmid (n+1). \end{cases}$$

Here $x_1$ denotes the class $[x_1]$ which equals the class $[x_2]$ since $d_2(v) = x_1 - x_2$. The generators $\sigma$ and $\overline{\sigma}$ are represented by $v$ multiplied by explicit polynomials in $x_1$ and $x_2$. The bidegrees are $\|\sigma\| = (2n-2, 1)$ and $\|\overline{\sigma}\| = (2n, 1)$.

The cohomology class $\pi_2^*(x_1)$, which equals $\pi_2^*(x_2)$ by Proposition 7.7 as $p \mid q$, represents $x_1$ in the spectral sequence. Since the complementary degree of $x_1$ is zero, the algebra structure of the spectral sequence gives us that $\pi_2^*(x_1)^{n+1} = 0$ for $p \mid (n+1)$ and $\pi_2^*(x_1)^n = 0$ for $p \nmid (n+1)$. By the left square in the diagram above, we get that the cohomology class $\pi_1^*(u)$ represents $w$ in the spectral sequence.

For $p \mid (n+1)$, $\sigma$ defines uniquely an unfiltered cohomology class since we have $E_\infty^{2n-1,0} = 0$. This class has $\sigma^2 = 0$ in the filtered quotient but since $E_\infty^{4n-3,1} = E_\infty^{4n-2,0} = 0$ this is also true in the actual cohomology ring. Similarly, for $p \nmid (n+1)$, $\overline{\sigma}$ defines uniquely an unfiltered cohomology class with $\overline{\sigma}^2 = 0$ since $E_\infty^{2n+1,0} = 0$ and $E_\infty^{4n+1,1} = E_\infty^{4n+2,0} = 0$.

Thus for $p \mid (n+1)$ we have a homomorphism of graded rings as follows:

$$\mathbb{F}_p[u, x, \sigma]/(x^{n+1}, \sigma^2) \to H_{\mathbb{T}}^*(\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}); \mathbb{F}_p);$$
$$u \mapsto \pi_1^*(u), \quad x \mapsto \pi_2^*(x_1) = \pi_2^*(x_2), \quad \sigma \mapsto \sigma$$

The homomorphism induces isomorphisms on associated graded objects, and therefore it is an isomorphism of rings. By this isomorphism and Proposition 7.7, we have that $pr_i^*(1 \otimes x) = \pi_2^*(x_1) = \pi_2^*(x_2) = x$ and $pr_i^*(u \otimes 1) = \pi_1^*(u) = u$, as desired. Similarly for $p \nmid (n+1)$.

THEOREM 7.9. *Let $n$, $q$ and $r$ be integers with $n > 1$ and $q > 0$. Let $p$ be a prime. Assume that $r = q \bmod 2$. Define two polynomials*

$$P(x_1, x_2) = \left(1 + \frac{r+q}{2q}(x_1 - x_2)\right)\left(1 + \frac{r-q}{2q}(x_1 - x_2)\right),$$
$$R(u) = \left(1 + \frac{r+q}{2}u\right)\left(1 + \frac{r-q}{2}u\right).$$

*In mod $p$ cohomology, we have total Chern classes as follows: if $p \nmid q$,*

$$c((\nu_{r,q})_{h\mathbb{T}}) = \frac{\left(1 + \frac{r+q}{2q}(x_1 - x_2) - x_1\right)^{n+1}}{P(x_1, x_2)},$$
$$c((\overline{\nu}_{r,q})_{h\mathbb{T}}) = \frac{\left(1 + \frac{r+q}{2q}(x_1 - x_2) + x_2\right)^{n+1}}{P(x_1, x_2)}$$

*and if $p \mid q$,*

$$c((\nu_{r,q})_{h\mathbb{T}}) = \frac{\left(1 + \frac{r+q}{2}u - x\right)^{n+1}}{R(u)}, \quad c((\overline{\nu}_{r,q})_{h\mathbb{T}}) = \frac{\left(1 + \frac{r+q}{2}u + x\right)^{n+1}}{R(u)}.$$

PROOF. Put $s_i = \frac{1}{2}(r + (-1)^{i+1}q)$, for $i = 1, 2$. By Proposition 7.1 and Remark 7.2 we have that

$$c_1(\overline{\gamma}_1(s_i)_{h\mathbb{T}}) = s_i u \otimes 1 - 1 \otimes x, \quad c_1(\epsilon^1(s_i)_{h\mathbb{T}}) = s_i u \otimes 1.$$

Assume that $p \nmid q$. From the pullbacks in Proposition 6.4 and from Theorem 7.8, we get first Chern classes

$$c_1\big(((L_0)_{r,q})_{h\mathbb{T}}\big) = \frac{s_i}{q}(x_1 - x_2) - x_i, \quad c_1\big(((L_i)_{r,q})_{h\mathbb{T}}\big) = \frac{s_i}{q}(x_1 - x_2).$$

Note that since $s_1/q(x_1 - x_2) - x_1 = s_2/q(x_1 - x_2) - x_2$ there is no contradiction in the first equation. By the direct sum decomposition in Theorem 6.3, the formula for the total Chern class of $(\nu_{r,q})_{h\mathbb{T}}$ follows. By a similar argument, we get the formula for the total Chern class of $(\overline{\nu}_{r,q})_{h\mathbb{T}}$.

Assume that $p \mid q$. In this case Proposition 6.4 and Theorem 7.8 give us first Chern classes $s_i u - x$ and $s_i u$ respectively, and via Theorem 6.3, the formula for the total Chern class of $(\nu_{r,q})_{h\mathbb{T}}$ follows. Similarly for $(\overline{\nu}_{r,q})_{h\mathbb{T}}$.

We can now prove our second main result regarding the bundles $\mu_q^- \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^n)$.

THEOREM 7.10. *Let $n$ and $q$ be integers with $n > 1$ and $q > 0$. Let $p$ be a prime. In cohomology with mod $p$ coefficients, we have total Chern classes as follows: for $p \nmid q$,*

$$c((\mu_q^-)_{h\mathbb{T}}) = \prod_{0<s<q} \left(1 + \frac{s}{q}(x_1 - x_2)\right) \cdot \prod_{\substack{0<r<q \\ r=q \bmod 2}} \frac{\left(\left(1 + \frac{r+q}{2q}(x_1 - x_2) - x_1\right)\left(1 + \frac{r+q}{2q}(x_1 - x_2) + x_2\right)\right)^{n+1}}{\left(\left(1 + \frac{r+q}{2q}(x_1 - x_2)\right)\left(1 + \frac{r-q}{2q}(x_1 - x_2)\right)\right)^2}.$$

*For $p \mid q$,*

$$c((\mu_q^-)_{h\mathbb{T}}) = \prod_{0<s<q} (1 + su) \prod_{\substack{0<r<q \\ r=q \bmod 2}} \frac{\left(\left(1 + \frac{r+q}{2}u - x\right)\left(1 + \frac{r+q}{2}u + x\right)\right)^{n+1}}{\left(\left(1 + \frac{r+q}{2}u\right)\left(1 + \frac{r-q}{2}u\right)\right)^2}.$$

PROOF. We use the direct sum decomposition from Theorem 5.10 which also gives a direct sum decomposition after forming $\mathbb{T}$-homotopy orbit bundles according to Proposition 6.5.

The bundle $\epsilon_q(\mathbb{R})_{h\mathbb{T}}$ is trivial, so its Chern classes are zero. The $\mathbb{T}$-vector bundle $\epsilon_q(\mathbb{C}(s))$ is the pullback of $\mathbb{C}\mathrm{P}^n \times \mathbb{C}(s) \longrightarrow \mathbb{C}\mathrm{P}^n$ along $pr_i\colon \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \longrightarrow \mathbb{C}\mathrm{P}^n$ both for $i = 1$ and $i = 2$. So by Proposition 7.1 and Theorem 7.8, we have

$$c_1(\epsilon_q(\mathbb{C}(s))_{h\mathbb{T}}) = pr_i^*(su \otimes 1) = \begin{cases} \frac{s}{q}(x_1 - x_2), & p \nmid q, \\ su, & p \mid q. \end{cases}$$

Theorem 7.9 above gives us the Chern classes of the remaining summands.

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