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PRESENTATIONS OF RINGS WITH A CHAIN OF SEMIDUALIZING MODULES

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Abstract

Inspired by Jorgensen et al., it is proved that if a Cohen-Macaulay local ring R with dualizing module admits a suitable chain of semidualizing R-modules of length n, then $R \cong Q/(I_1 + \cdots + I_n)$ for some Gorenstein ring Q and ideals $I_1, \ldots, I_n$ of Q; and, for each $\Lambda \subseteq [n]$, the ring $Q/\big(\sum_{\ell\in\Lambda} I_\ell\big)$ has some interesting cohomological properties. This extends the result of Jorgensen et al., and also of Foxby and Reiten.

1. Introduction

Throughout R is a commutative noetherian local ring. Foxby [4], Vasconcelos [17] and Golod [8] independently initiated the study of semidualizing modules. A finite (i.e. finitely generated) R-module C is called *semidualizing* if the natural homothety map $\chi^R_C\colon R \longrightarrow \operatorname{Hom}_R(C, C)$ is an isomorphism and $\operatorname{Ext}^{\geqslant 1}_R(C, C) = 0$ (see [10, Definition 1.1]). Examples of semidualizing R-modules include R itself and a dualizing R-module when one exists. The set of all isomorphism classes of semidualizing R-modules is denoted by $\mathfrak{G}_0(R)$, and the isomorphism class of a semidualizing R-module C is denoted $[C]$. The set $\mathfrak{G}_0(R)$ has caught the attention of several authors; see, for example [6], [3], [12] and [15]. In [3], Christensen and Sather-Wagstaff show that $\mathfrak{G}_0(R)$ is finite when R is Cohen-Macaulay and equicharacteristic. Then Nasseh and Sather-Wagstaff, in [12], settle the general assertion that $\mathfrak{G}_0(R)$ is finite. Also, in [15], Sather-Wagstaff studies the cardinality of $\mathfrak{G}_0(R)$.

Each semidualizing R-module C gives rise to a notion of reflexivity for finite R-modules. For instance, each finite projective R-module is totally C-reflexive. For semidualizing R-modules C and B, we write $[C] \trianglelefteq [B]$ whenever B is totally C-reflexive. In [7], Gerko defines chains in $\mathfrak{G}_0(R)$. A *chain* in $\mathfrak{G}_0(R)$ is a sequence $[C_n] \trianglelefteq \cdots \trianglelefteq [C_1] \trianglelefteq [C_0]$, and such a chain has length n if $[C_i] \neq [C_j]$, whenever $i \neq j$. In [15], Sather-Wagstaff uses

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the length of chains in $\mathfrak{G}_0(R)$ to provide a lower bound for the cardinality of $\mathfrak{G}_0(R)$.

It is well-known that a Cohen-Macaulay ring which is homomorphic image of a Gorenstein local ring, admits a dualizing module (see [16, Theorem 3.9]). Then Foxby [4] and Reiten [13], independently, prove the converse. Recently Jorgensen et al. [11], characterize the Cohen-Macaulay local rings which admit dualizing modules and non-trivial semidualizing modules (i.e. neither free nor dualizing).

In this paper, we are interested in characterization of Cohen-Macaulay rings R which admit a dualizing module and a certain chain in $\mathfrak{G}_0(R)$. We prove that, when a Cohen-Macaulay ring R with dualizing module has a *suitable chain* in $\mathfrak{G}_0(R)$ (see Definition 3.1) of length n, then there exist a Gorenstein ring Q and ideals $I_1, \dots, I_n$ of Q such that $R \cong Q/(I_1 + \cdots + I_n)$ and, for each $\Lambda \subseteq [n] = \{1, \dots, n\}$, the ring $Q/\big(\sum_{\ell \in \Lambda} I_\ell\big)$ has certain homological and cohomological properties (see Theorem 3.9). Note that, this result gives the result of Jorgensen et al. when $n = 2$ and the result of Foxby and Reiten in the case $n = 1$. We prove a partial converse of Theorem 3.9 in Propositions 3.15 and 3.16.

2. Preliminaries

This section contains definitions and background material.

DEFINITION 2.1 ([10, Definition 2.7] and [14, Theorem 5.2.3 and Definition 6.1.2]). Let C be a semidualizing R-module. A finite R-module M is *totally C-reflexive* when it satisfies the following conditions:

(i) the natural homomorphism $\delta^C_M \colon M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$ is an isomorphism, and

(ii) $\operatorname{Ext}^{\geqslant 1}_R(M, C) = 0 = \operatorname{Ext}^{\geqslant 1}_R(\operatorname{Hom}_R(M, C), C)$.

A totally R-reflexive is referred to as totally reflexive. The G_C-*dimension* of a finite R-module M, denoted $\mathrm{G}_C\text{-}\dim_R(M)$, is defined as

$$\mathrm{G}_C\text{-}\dim_R(M) = \inf\left\{ n \geqslant 0 \;\middle|\; \begin{array}{l} \text{there is an exact sequence of } R\text{-modules} \\ 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \\ \text{such that each } G_i \text{ is totally } C\text{-reflexive} \end{array} \right\}.$$

REMARK 2.2 ([2, Theorem 6.1]). Let S be a Cohen-Macaulay local ring equipped with a module-finite local ring homomorphism $\tau \colon R \to S$ such that R is Cohen-Macaulay. Assume that C is a semidualizing R-module. Then $\mathrm{G}_C\text{-}\dim_R(S) < \infty$ if and only if there exists an integer $g \geqslant 0$ such that $\operatorname{Ext}^i_R(S, C) = 0$, for all $i \neq g$, and $\operatorname{Ext}^g_R(S, C)$ is a semidualizing S-module. When these conditions hold, one has $g = \mathrm{G}_C\text{-}\dim_R(S)$.

DEFINITION 2.3 (The order $\trianglelefteq$ on $\mathfrak{G}_0(R)$). For $[B], [C] \in \mathfrak{G}_0(R)$, write $[C] \trianglelefteq [B]$ when B is totally C-reflexive (see, e.g., [15]). This relation is reflexive and antisymmetric [5, Lemma 3.2], but it is not known whether it is transitive in general. Also, write $[C] \lhd [B]$ when $[C] \trianglelefteq [B]$ and $[C] \neq [B]$. For a semidualizing C, set

$$\mathfrak{G}_C(R) = \big\{[B] \in \mathfrak{G}_0(R) \;\big|\; [C] \trianglelefteq [B]\big\}.$$

In the case D is a dualizing R-module, one has $[D] \trianglelefteq [B]$ for any semidualizing R-module B, by [9, (V.2.1)], and so $\mathfrak{G}_D(R) = \mathfrak{G}_0(R)$.

If $[C] \trianglelefteq [B]$, then $\operatorname{Hom}_R(B, C)$ is a semidualizing and $[C] \trianglelefteq [\operatorname{Hom}_R(B, C)]$ ([2, Theorem 2.11]). Moreover, if A is another semidualizing R-module with $[C] \trianglelefteq [A]$, then $[B] \trianglelefteq [A]$ if and only if $[\operatorname{Hom}_R(A, C)] \trianglelefteq [\operatorname{Hom}_R(B, C)]$ ([5, Proposition 3.9]).

THEOREM 2.4 ([7, Theorem 3.1]). *Let B and C be two semidualizing R-modules such that $[C] \trianglelefteq [B]$. Assume that M is an R-module which is both totally B-reflexive and totally C-reflexive, then the composition map*

$$\varphi\colon \operatorname{Hom}_R(M, B) \otimes_R \operatorname{Hom}_R(B, C) \longrightarrow \operatorname{Hom}_R(M, C)$$

is an isomorphism.

COROLLARY 2.5 ([7, Corollary 3.3]). *If $[C_n] \trianglelefteq \cdots \trianglelefteq [C_1] \trianglelefteq [C_0]$ is a chain in $\mathfrak{G}_0(R)$, then one gets*

$$C_n \cong C_0 \otimes_R \operatorname{Hom}_R(C_0, C_1) \otimes_R \cdots \otimes_R \operatorname{Hom}_R(C_{n-1}, C_n).$$

Assume that $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ is a chain in $\mathfrak{G}_0(R)$. For each $i \in [n]$, set $B_i = \operatorname{Hom}_R(C_{i-1}, C_i)$. For each sequence of integers $\mathbf{i} = \{i_1, \ldots, i_j\}$ with $j \geqslant 1$ and $1 \leqslant i_1 < \cdots < i_j \leqslant n$, set $B_{\mathbf{i}} = B_{i_1} \otimes_R \cdots \otimes_R B_{i_j}$. ($B_{\{i_1\}} = B_{i_1}$ and set $B_\emptyset = C_0$.)

In order to facilitate the discussion, we list some results from [15]. We first recall the following definition.

DEFINITION 2.6. Let C be a semidualizing R-module. The *Auslander class* $\mathscr{A}_C(R)$ with respect to C is the class of all R-modules M satisfying the following conditions:

(1) the natural map $\gamma_M^C\colon M \longrightarrow \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism,

(2) $\operatorname{Tor}^R_{\geqslant 1}(C, M) = 0 = \operatorname{Ext}_R^{\geqslant 1}(C, C \otimes_R M)$.

PROPOSITION 2.7. *Assume that* $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ *is a chain in* $\mathfrak{G}_0(R)$ *such that* $\mathfrak{G}_{C_1}(R) \subseteq \mathfrak{G}_{C_2}(R) \subseteq \cdots \subseteq \mathfrak{G}_{C_n}(R)$.

(1) [15, Lemma 4.3] *For each* i, p *with* $1 \leqslant i \leqslant i + p \leqslant n$,

$$B_{\{i,i+1,\ldots,i+p\}} \cong \mathrm{Hom}_R(C_{i-1}, C_{i+p}).$$

(2) [15, Lemma 4.4] *If* $1 \leqslant i < j - 1 \leqslant n - 1$, *then*

$$B_{\{i,j\}} \cong \mathrm{Hom}_R(\mathrm{Hom}_R(B_i, C_{j-1}), C_j).$$

(3) [15, Lemma 4.5] *For each sequence* $\mathbf{i} = \{i_1, \ldots, i_j\} \subseteq [n]$, *the* R-*module* $B_{\mathbf{i}}$ *is a semidualizing.*

(4) [15, Lemma 4.6] *If* $\mathbf{i} = \{i_1, \ldots, i_j\} \subseteq [n]$ *and* $\mathbf{s} = \{s_1, \ldots, s_t\} \subseteq [n]$ *are two sequences with* $\mathbf{s} \subseteq \mathbf{i}$, *then* $[B_{\mathbf{i}}] \trianglelefteq [B_{\mathbf{s}}]$ *and* $\mathrm{Hom}_R(B_{\mathbf{s}}, B_{\mathbf{i}}) \cong B_{\mathbf{i}\setminus\mathbf{s}}$.

(5) [15, Theorem 4.11] *If* $\mathbf{i} = \{i_1, \ldots, i_j\} \subseteq [n]$ *and* $\mathbf{s} = \{s_1, \ldots, s_t\} \subseteq [n]$ *are two sequences, then the following conditions are equivalent:*
   - (a) $B_{\mathbf{i}} \in \mathscr{A}_{B_{\mathbf{s}}}(R)$,
   - (b) $B_{\mathbf{s}} \in \mathscr{A}_{B_{\mathbf{i}}}(R)$,
   - (c) *the* R-*module* $B_{\mathbf{i}} \otimes_R B_{\mathbf{s}}$ *is semidualizing,*
   - (d) $\mathbf{i} \cap \mathbf{s} = \emptyset$.

At the end of this section we recall the definition of trivial extension ring. Note that this notion is the main key in the proof of the converse of Sharp's result [16], which is given by Foxby [4] and Reiten [13].

DEFINITION 2.8. For an R-module M, the *trivial extension* of R by M is the ring $R \ltimes M$, described as follows. As an R-module, we have $R \ltimes M = R \oplus M$. The multiplication is defined by $(r, m)(r', m') = (rr', rm' + r'm)$. Note that the composition $R \to R \ltimes M \to R$ of the natural homomorphisms is the identity map of R.

Note that, for a semidualizing R-module C, the trivial extension ring $R \ltimes C$ is a commutative noetherian local ring. If R is Cohen-Macaulay then $R \ltimes C$ is Cohen-Macaulay too. For more information about the trivial extension rings one may see, e.g., [11, Section 2].

3. Results

This section is devoted to the main result, Theorem 3.9, which extends the results of Jorgensen et al. [11, Theorem 3.2] and of Foxby [4] and Reiten [13].

For a semidualizing R-module C, set $(-)^{\dagger_C} = \mathrm{Hom}_R(-, C)$. The following notations are taken from [15].

DEFINITION 3.1. Let $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ be a chain in $\mathfrak{G}_0(R)$ of length n. For each sequence of integers $\mathbf{i} = \{i_1, \ldots, i_j\}$ such that $j \geqslant 0$ and $1 \leqslant i_1 < \ldots < i_j \leqslant n$, set $C_{\mathbf{i}} = C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \cdots \dagger_{C_{i_j}}}$. (When $j = 0$, set $C_{\mathbf{i}} = C_\emptyset = C_0$.)

We say that the above chain is *suitable* if $C_0 = R$ and $C_{\mathbf{i}}$ is totally C_t-reflexive, for all $\mathbf{i}$ and t with $i_j \leqslant t \leqslant n$.

Note that if $[C_n] \lhd \cdots \lhd [C_1] \lhd [R]$ is a suitable chain, then $C_{\mathbf{i}}$ is a semidualizing R-module for each $\mathbf{i} \subseteq [n]$. Also, for each sequence of integers $\{x_1, \ldots, x_m\}$ with $1 \leqslant x_1 < \cdots < x_m \leqslant n$, the sequence $[C_{x_m}] \lhd \cdots \lhd [C_{x_1}] \lhd [R]$ is a suitable chain in $\mathfrak{G}_0(R)$ of length m.

Sather-Wagstaff, in [15, Theorem 3.3], proves that if $\mathfrak{G}_0(R)$ admits a chain $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ such that $\mathfrak{G}_{C_0}(R) \subseteq \mathfrak{G}_{C_1}(R) \subseteq \cdots \subseteq \mathfrak{G}_{C_n}(R)$, then $|\mathfrak{G}_0(R)| \geqslant 2^n$. Indeed, the classes $[C_{\mathbf{i}}]$, which are parameterized by the allowable sequences $\mathbf{i}$, are precisely the 2^n classes constructed in the proof of [15, Theorem 3.3].

THEOREM 3.2 ([15, Theorem 4.7]). *Let $\mathfrak{G}_0(R)$ admit a chain $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ such that $\mathfrak{G}_{C_1}(R) \subseteq \mathfrak{G}_{C_2}(R) \subseteq \cdots \subseteq \mathfrak{G}_{C_n}(R)$. If $C_0 = R$, then the R-modules $B_{\mathbf{i}}$ are precisely the 2^n semidualizing modules constructed in [15, Theorem 3.3].*

REMARK 3.3. In Proposition 2.7 and Theorem 3.2, if we replace the assumption of existence of a chain $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ in $\mathfrak{G}_0(R)$ such that $\mathfrak{G}_{C_1}(R) \subseteq \mathfrak{G}_{C_2}(R) \subseteq \cdots \subseteq \mathfrak{G}_{C_n}(R)$ by the existence of a suitable chain, then the assertions hold true as well.

The next lemma and proposition give us sufficient tools to treat Theorem 3.9.

LEMMA 3.4. *Assume that R admits a suitable chain $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0] = [R]$ in $\mathfrak{G}_0(R)$. Then for any $k \in [n]$, there exists a suitable chain*

$$[C_n] \lhd \cdots \lhd [C_{k+1}] \lhd [C_k] \lhd [C_1^{\dagger_{C_k}}] \lhd \cdots \lhd [C_{k-2}^{\dagger_{C_k}}] \lhd [C_{k-1}^{\dagger_{C_k}}] \lhd [R] \quad (1)$$

in $\mathfrak{G}_0(R)$ of length n.

PROOF. For $i, j, 0 \leqslant j < i \leqslant k$, as $[C_i] \lhd [C_j]$ one has $[C_j^{\dagger_{C_k}}] \lhd [C_i^{\dagger_{C_k}}]$. As $[C_k] \neq [C_i^{\dagger_{C_k}}]$, one gets $[C_t] \lhd [C_i^{\dagger_{C_k}}]$ for each $t, k \leqslant t \leqslant n$. Thus (1) is a chain in $\mathfrak{G}_0(R)$ of length n.

Next, we show that (1) is a suitable chain. For $r, t \in \{0, 1, \ldots, n\}$ and a sequence $\{x_1, \ldots, x_m\}$ of integers with $r \leqslant x_1 < \cdots < x_m \leqslant t$, repeated use of Theorem 2.4 implies

$$C_r^{\dagger_{C_t}} \cong C_r^{\dagger_{C_{x_1}}} \otimes_R C_{x_1}^{\dagger_{C_{x_2}}} \otimes_R \cdots \otimes_R C_{x_m}^{\dagger_{C_t}}.$$

For each r, $0 < r < k$, set $C'_r = C_r^{\dagger_{C_k}}$. If $\mathbf{i} = \{i_1, \ldots, i_j\}$ and $\mathbf{u} = \{u_1, \ldots, u_s\}$ are sequences of integers such that $j, s \geqslant 0$ and $1 \leqslant i_j < \cdots < i_1 < k \leqslant u_1 < \cdots < u_s \leqslant n$, then we set

$$C_{\mathbf{i},\mathbf{u}} = C_0^{\dagger_{C'_{i_1}} \cdots \dagger_{C'_{i_j}} \dagger_{C_{u_1}} \cdots \dagger_{C_{u_s}}}.$$

When $s = 0$ (resp., $j = 0$ or $j = 0 = s$), we have $C_{\mathbf{i},\mathbf{u}} = C_{\mathbf{i},\emptyset}$ (resp., $C_{\mathbf{i},\mathbf{u}} = C_{\emptyset,\mathbf{u}}$ or $C_{\mathbf{i},\mathbf{u}} = C_{\emptyset,\emptyset} = C_0$).

By Proposition 2.7(4) and Remark 3.3, one has $C_0^{\dagger_{C'_{i_1}} \dagger_{C'_{i_2}}} \cong \operatorname{Hom}_R\big(C_{i_1}^{\dagger_{C_k}}, C_{i_2}^{\dagger_{C_k}}\big) \cong C_{i_2}^{\dagger_{C_{i_1}}}$ and so $C_0^{\dagger_{C'_{i_1}} \dagger_{C'_{i_2}} \dagger_{C'_{i_3}}} \cong \operatorname{Hom}_R\big(C_{i_2}^{\dagger_{C_{i_1}}}, C_{i_3}^{\dagger_{C_k}}\big) \cong C_{i_3}^{\dagger_{C_{i_2}}} \otimes_R C_{i_1}^{\dagger_{C_k}}$. By proceeding in this way one obtains the following isomorphism

$$C_0^{\dagger_{C'_{i_1}} \cdots \dagger_{C'_{i_j}}} \cong \begin{cases} C_{i_j}^{\dagger_{C_{i_{j-1}}}} \otimes_R C_{i_{j-2}}^{\dagger_{C_{i_{j-3}}}} \otimes_R \cdots \otimes_R C_{i_2}^{\dagger_{C_{i_1}}}, & \text{if } j \text{ is even,} \\ C_{i_j}^{\dagger_{C_{i_{j-1}}}} \otimes_R C_{i_{j-2}}^{\dagger_{C_{i_{j-3}}}} \otimes_R \cdots \otimes_R C_{i_1}^{\dagger_{C_k}}, & \text{if } j \text{ is odd.} \end{cases} \tag{2}$$

Therefore, by Proposition 2.7(2) and Remark 3.3,

$$C_0^{\dagger_{C'_{i_1}} \cdots \dagger_{C'_{i_j}}} \cong \begin{cases} C_0^{\dagger_{C_{i_j}} \cdots \dagger_{C_{i_1}}}, & \text{if } j \text{ is even,} \\ C_0^{\dagger_{C_{i_j}} \cdots \dagger_{C_{i_1}} \dagger_{C_k}}, & \text{if } j \text{ is odd,} \end{cases}$$

and thus

$$C_{\mathbf{i},\mathbf{u}} \cong \begin{cases} C_0^{\dagger_{C_{i_j}} \cdots \dagger_{C_{i_1}} \dagger_{C_{u_1}} \cdots \dagger_{C_{u_s}}}, & \text{if } j \text{ is even,} \\ C_0^{\dagger_{C_{i_j}} \cdots \dagger_{C_{i_1}} \dagger_{C_k} \dagger_{C_{u_1}} \cdots \dagger_{C_{u_s}}}, & \text{if } j \text{ is odd.} \end{cases}$$

Hence, by assumption, $[C_t] \trianglelefteq [C_{\mathbf{i},\mathbf{u}}]$ for all t, $t \geqslant u_s$. If $s = 0$, then $C_{\mathbf{i},\mathbf{u}} = C_{\mathbf{i},\emptyset} = C_0^{\dagger_{C'_{i_1}} \cdots \dagger_{C'_{i_j}}}$.

On the other hand, for each ℓ, $1 \leqslant \ell \leqslant i_j$, we have

$$C_\ell^{\dagger_{C_k}} \cong C_\ell^{\dagger_{C_{i_j}}} \otimes_R C_{i_j}^{\dagger_{C_{i_{j-1}}}} \otimes_R \cdots \otimes_R C_{i_3}^{\dagger_{C_{i_2}}} \otimes_R C_{i_2}^{\dagger_{C_{i_1}}} \otimes_R C_{i_1}^{\dagger_{C_k}}.$$

Thus, by Proposition 2.7(4) and (2), $\big[C_\ell^{\dagger_{C_k}}\big] \trianglelefteq [C_{\mathbf{i},\mathbf{u}}]$. Hence the chain (1) is suitable.

REMARK 3.5. Let R be Cohen-Macaulay and $[C_n] \lhd \cdots \lhd [C_1] \lhd [C_0]$ be a suitable chain in $\mathfrak{G}_0(R)$. For any k, $1 \leqslant k \leqslant n$, set $R_k = R \ltimes C_{k-1}^{\dagger_{C_k}}$, the trivial extension of R by $C_{k-1}^{\dagger_{C_k}}$. Then R_k is totally $C_\ell^{\dagger_{C_k}}$-reflexive and totally C_t-reflexive R-module for all ℓ, t with $1 \leqslant \ell < k \leqslant t \leqslant n$. Set

$$C_\ell^{(k)} = \begin{cases} \operatorname{Hom}_R\big(R_k, C_{k-1-\ell}^{\dagger_{C_k}}\big), & \text{if } 0 \leqslant \ell < k-1, \\ \operatorname{Hom}_R\big(R_k, C_{\ell+1}\big), & \text{if } k-1 \leqslant \ell \leqslant n-1. \end{cases}$$

Then, by Remark 2.2, $C_\ell^{(k)}$ is a semidualizing R_k-module for all ℓ, $0 \leqslant \ell \leqslant n-1$.

PROPOSITION 3.6. *Under the hypotheses of Remark 3.5, for all k,* $1 \leqslant k \leqslant n$,

$$[C_{n-1}^{(k)}] \lhd \cdots \lhd [C_1^{(k)}] \lhd [R_k]$$

is a suitable chain in $\mathfrak{G}_0(R_k)$ *of length* $n-1$.

PROOF. Let $k \in [n]$. For integers a, b with $a \neq b$ and $0 \leqslant a, b \leqslant n-1$, we observe that $[C_a^{(k)}] \neq [C_b^{(k)}]$. Indeed, we consider the three cases $0 \leqslant a, b < k-1$, $0 \leqslant a < k-1 \leqslant b \leqslant n-1$, and $k-1 \leqslant a, b \leqslant n-1$. We only discuss the first case. The other cases are treated in a similar way. For $0 \leqslant a, b < k-1$, if $[C_a^{(k)}] = [C_b^{(k)}]$, then $\operatorname{Hom}_R\big(R_k, C_{k-1-a}^{\dagger_{C_k}}\big) \cong \operatorname{Hom}_R\big(R_k, C_{k-1-b}^{\dagger_{C_k}}\big)$ and so $\operatorname{Hom}_{R_k}\big(R, \operatorname{Hom}_R\big(R_k, C_{k-1-a}^{\dagger_{C_k}}\big)\big) \cong \operatorname{Hom}_{R_k}\big(R, \operatorname{Hom}_R\big(R_k, C_{k-1-b}^{\dagger_{C_k}}\big)\big)$. Thus, by adjointness, $C_{k-1-a}^{\dagger_{C_k}} \cong C_{k-1-b}^{\dagger_{C_k}}$, which contradicts with (1) in Lemma 3.4.

In order to proceed with the proof, for an R_k-module M, we invent the symbol $(-)^{\dagger^k_M} = \operatorname{Hom}_{R_k}(-, M)$. Note that, for R_k-modules $M_1, \ldots, M_t$, we have

$$(-)^{\dagger^k_{M_1}\dagger^k_{M_2}\cdots\dagger^k_{M_t}} = \bigg(\Big(\big((-)^{\dagger^k_{M_1}}\big)^{\dagger^k_{M_2}}\Big)^{\cdots}\bigg)^{\dagger^k_{M_t}} = \operatorname{Hom}_{R_k}\big((-)^{\dagger^k_{M_1}\dagger^k_{M_2}\cdots\dagger^k_{M_{t-1}}}, M_t\big).$$

For two sequences of integers $\mathbf{p} = \{p_1, \ldots, p_r\}$ and $\mathbf{q} = \{q_1, \ldots, q_s\}$ such that $r, s \geqslant 0$ and $0 < p_1 < \cdots < p_r < k-1 \leqslant q_1 < \cdots < q_s \leqslant n-1$, set

$$C_{\mathbf{p},\mathbf{q}}^{(k)} = R_k^{\dagger^k_{C_{p_1}^{(k)}}\cdots\dagger^k_{C_{p_r}^{(k)}}\dagger^k_{C_{q_1}^{(k)}}\cdots\dagger^k_{C_{q_s}^{(k)}}}.$$

Therefore one gets the following R-module isomorphisms

$$
\begin{aligned}
C^{(k)}_{\mathbf{p},\mathbf{q}} &= \operatorname{Hom}_{R_k}(\ldots \operatorname{Hom}_{R_k}(\operatorname{Hom}_{R_k}(\\
&\qquad \ldots \operatorname{Hom}_{R_k}(R_k, C^{(k)}_{p_1}) \ldots, C^{(k)}_{p_r}), C^{(k)}_{q_1}) \ldots, C^{(k)}_{q_s}) \\
&\cong \operatorname{Hom}_{R}(\ldots \operatorname{Hom}_{R}(\operatorname{Hom}_{R}(\\
&\qquad \ldots \operatorname{Hom}_{R}(R_k, C^{\dagger_{C_k}}_{k-1-p_1}) \ldots, C^{\dagger_{C_k}}_{k-1-p_r}), C_{q_1+1}) \ldots, C_{q_s+1}) \\
&\cong R^{\dagger_{C'_{k-1-p_1}} \cdots \dagger_{C'_{k-1-p_r}} \dagger_{C_{q_1+1}} \cdots \dagger_{C_{q_s+1}}} \oplus R^{\dagger_{C'_{k-1}} \dagger_{C'_{k-1-p_1}} \cdots \dagger_{C'_{k-1-p_r}} \dagger_{C_{q_1+1}} \cdots \dagger_{C_{q_s+1}}} \\
&= C_{\mathbf{i},\mathbf{u}} \oplus C_{\mathbf{i}',\mathbf{u}},
\end{aligned}
$$

where $\mathbf{i} = \{k-1-p_1, \ldots, k-1-p_r\}$, $\mathbf{i}' = \{k-1, k-1-p_1, \ldots, k-1-p_r\}$, $\mathbf{u} = \{q_1+1, \ldots, q_s+1\}$, $C'_\ell = C^{\dagger_{C_k}}_\ell$, for all $0 < \ell < k$, and $C_{\mathbf{i},\mathbf{u}}$ and $C_{\mathbf{i}',\mathbf{u}}$ are as in the proof of Lemma 3.4.

As $[C_{t+1}] \trianglelefteq [C_{\mathbf{i},\mathbf{u}}]$ and $[C_{t+1}] \trianglelefteq [C_{\mathbf{i}',\mathbf{u}}]$ in $\mathfrak{G}_0(R)$ for all t, $q_s \leqslant t \leqslant n-1$, one gets $[C^{(k)}_t] \trianglelefteq [C^{(k)}_{\mathbf{p},\mathbf{q}}]$ in $\mathfrak{G}_0(R_k)$, by [2, Theorem 6.5]. When $s = 0$ we have $C^{(k)}_{\mathbf{p},\mathbf{q}} = C^{(k)}_{\mathbf{p},\emptyset} \cong C_{\mathbf{i},\emptyset} \oplus C_{\mathbf{i}',\emptyset}$. By Lemma 3.4, for all m, $p_r \leqslant m < k-1$, one has $\left[C^{\dagger_{C_k}}_{k-1-m}\right] \trianglelefteq [C_{\mathbf{i},\emptyset}]$ and $\left[C^{\dagger_{C_k}}_{k-1-m}\right] \trianglelefteq [C_{\mathbf{i}',\emptyset}]$ in $\mathfrak{G}_0(R)$. Thus, by [2, Theorem 6.5], one gets $[C^{(k)}_m] \trianglelefteq [C^{(k)}_{\mathbf{p},\emptyset}]$ in $\mathfrak{G}_0(R_k)$. Hence $[C^{(k)}_{n-1}] \lhd \cdots \lhd [C^{(k)}_1] \lhd [R_k]$ is a suitable chain in $\mathfrak{G}_0(R_k)$ of length $n-1$.

To state our main result, we recall the definitions of Tate homology and Tate cohomology (see [1] and [11] for more details).

Definition 3.7. Let M be a finite R-module. A *Tate resolution* of M is a diagram $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$, where π is an R-projective resolution of M, $\mathbf{T}$ is an exact complex of projectives such that $\operatorname{Hom}_R(T, R)$ is exact, ϑ is a morphism, and ϑ_i is isomorphism for all $i \gg 0$.

By [1, Theorem 3.1], a finite R-module has finite G-dimension if and only if it admits a Tate resolution.

Definition 3.8. Let M be a finite R-module of finite G-dimension, and let $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$ be a Tate resolution of M. For each integer i and each R-module N, the ith *Tate homology* and *Tate cohomology* modules are

$$\widehat{\operatorname{Tor}}{}^R_i(M, N) = \mathrm{H}_i(\mathbf{T} \otimes_R N), \quad \widehat{\operatorname{Ext}}{}^i_R(M, N) = \mathrm{H}_{-i}(\operatorname{Hom}_R(\mathbf{T}, N)).$$

Theorem 3.9. *Let R be a Cohen-Macaulay ring with a dualizing module D. Assume that R admits a suitable chain $[C_n] \lhd \cdots \lhd [C_1] \lhd [R]$ in $\mathfrak{G}_0(R)$ and that $C_n \cong D$. Then there exist a Gorenstein local ring Q and ideals $I_1, \ldots, I_n$*

of Q, which satisfy the conditions below. In this situation, for each $\Lambda \subseteq [n]$, set $R_\Lambda = Q/(\sum_{\ell\in\Lambda} I_\ell)$, in particular $R_\emptyset = Q$.

(1) *There is a ring isomorphism $R \cong Q/(I_1 + \cdots + I_n)$.*

(2) *For each $\Lambda \subseteq [n]$ with $\Lambda \neq \emptyset$, the ring R_Λ is non-Gorenstein Cohen-Macaulay with a dualizing module.*

(3) *For each $\Lambda \subseteq [n]$ with $\Lambda \neq \emptyset$, we have $\bigcap_{\ell\in\Lambda} I_\ell = \prod_{\ell\in\Lambda} I_\ell$.*

(4) *For subsets Λ, Γ of $[n]$ with $\Gamma \subsetneq \Lambda$, we have $\operatorname{G-dim}_{R_\Gamma} R_\Lambda = 0$, and $\operatorname{Hom}_{R_\Gamma}(R_\Lambda, R_\Gamma)$ is a non-free semidualizing R_Λ-module.*

(5) *For subsets Λ, Γ of $[n]$ with $\Lambda \neq \Gamma$, the module $\operatorname{Hom}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma)$ is not cyclic and*

$$\operatorname{Ext}^{\geqslant 1}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) = 0 = \operatorname{Tor}^{R_{\Lambda\cap\Gamma}}_{\geqslant 1}(R_\Lambda, R_\Gamma).$$

(6) *For subsets Λ, Γ of $[n]$ with $|\Lambda \setminus \Gamma| = 1$, we have*

$$\widehat{\operatorname{Ext}}^{i}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) = 0 = \widehat{\operatorname{Tor}}^{R_{\Lambda\cap\Gamma}}_{i}(R_\Lambda, R_\Gamma)$$

for all $i \in \mathbb{Z}$.

The ring Q is constructed as an iterated trivial extension of R. As an R-module, it has the form $Q = \bigoplus_{\mathbf{i}\subseteq[n]} B_\mathbf{i}$. The details are contained in the following construction.

Construction 3.10. We construct the ring Q by induction on n. We claim that the ring Q, as an R-module, has the form $Q = \bigoplus_{\mathbf{i}\subseteq[n]} B_\mathbf{i}$ and the ring structure on it is as follows: for two elements $(\alpha_\mathbf{i})_{\mathbf{i}\subseteq[n]}$ and $(\theta_\mathbf{i})_{\mathbf{i}\subseteq[n]}$ of Q,

$$(\alpha_\mathbf{i})_{\mathbf{i}\subseteq[n]}(\theta_\mathbf{i})_{\mathbf{i}\subseteq[n]} = (\sigma_\mathbf{i})_{\mathbf{i}\subseteq[n]}, \qquad \text{where} \quad \sigma_\mathbf{i} = \sum_{\substack{\mathbf{v}\subseteq\mathbf{i}\\ \mathbf{w}=\mathbf{i}\setminus\mathbf{v}}} \alpha_\mathbf{v}\cdot\theta_\mathbf{w}.$$

For $n = 1$, set $Q = R \ltimes C_1$ and $I_1 = 0 \oplus C_1$, which is the result of Foxby [4] and Reiten [13]. The case $n = 2$ is proved by Jorgensen et al. [11, Theorem 3.2]. They proved that the extension ring Q has the form $Q = R \oplus C_1 \oplus C_1^{\dagger_{C_2}} \oplus C_2$ as an R-module (i.e. $Q = B_\emptyset \oplus B_1 \oplus B_2 \oplus B_{\{1,2\}}$). Also the ring structure on Q is given by $(r, c, f, d)(r', c', f', d') = (rr', rc' + r'c, rf' + r'f, f'(c) + f(c') + rd' + r'd)$. The ideal I_ℓ, $\ell = 1, 2$, has the form $I_\ell = 0 \oplus 0 \oplus B_\ell \oplus B_{\{1,2\}}$.

Let $n > 2$. Take an element $k \in [n]$. By Proposition 3.6, the ring $R_k = R \ltimes C_{k-1}^{\dagger_{C_k}}$ has the suitable chain $[C_{n-1}^{(k)}] \lhd \cdots \lhd [C_1^{(k)}] \lhd [R_k]$ in $\mathfrak{G}_0(R_k)$ of length $n-1$. Note that $C_{n-1}^{(k)} = \operatorname{Hom}_R(R_k, C_n) \cong \operatorname{Hom}_R(R_k, D)$ is a dualizing R_k-module.

We set $B_i^{(k)} = \operatorname{Hom}_{R_k}(C_{i-1}^{(k)}, C_i^{(k)})$, $i = 1, \ldots, n-1$. For two sequences $\mathbf{p} = \{p_1, \ldots, p_r\}$, $\mathbf{q} = \{q_1, \ldots, q_s\}$ such that $r, s \geqslant 1$ and $1 \leqslant p_1 < \cdots < p_r < k-1 \leqslant q_1 < \cdots < q_s \leqslant n-1$, we set

$$B_{\mathbf{p},\mathbf{q}}^{(k)} = B_{p_1}^{(k)} \otimes_{R_k} \cdots \otimes_{R_k} B_{p_r}^{(k)} \otimes_{R_k} B_{q_1}^{(k)} \otimes_{R_k} \cdots \otimes_{R_k} B_{q_s}^{(k)}, \tag{3}$$

and

$$B_{\mathbf{p},\emptyset}^{(k)} = B_{p_1}^{(k)} \otimes_{R_k} \cdots \otimes_{R_k} B_{p_r}^{(k)}, \quad B_{\emptyset,\mathbf{q}}^{(k)} = B_{q_1}^{(k)} \otimes_{R_k} \cdots \otimes_{R_k} B_{q_s}^{(k)},$$

and

$$B_{\emptyset,\emptyset}^{(k)} = C_0^{(k)} = R_k.$$

By applying the induction hypothesis on R_k, there is an extension ring, say Q_k, which is Gorenstein local and, as an R_k-module, has the form

$$Q_k = \bigoplus_{\substack{\mathbf{p}\subseteq\{1,\ldots,k-2\}\\ \mathbf{q}\subseteq\{k-1,\ldots,n-1\}}} B_{\mathbf{p},\mathbf{q}}^{(k)}.$$

Moreover, the ring structure on Q_k is as follows: for $\phi = (\phi_{\mathbf{p},\mathbf{q}})_{\substack{\mathbf{p}\subseteq\{1,\ldots,k-2\},\\ \mathbf{q}\subseteq\{k-1,\ldots,n-1\}}}$ and $\varphi = (\varphi_{\mathbf{p},\mathbf{q}})_{\mathbf{p}\subseteq\{1,\ldots,k-2\},\, \mathbf{q}\subseteq\{k-1,\ldots,n-1\}}$ of Q_k

$$\phi\varphi = \psi = (\psi_{\mathbf{p},\mathbf{q}})_{\mathbf{p}\subseteq\{1,\ldots,k-2\},\, \mathbf{q}\subseteq\{k-1,\ldots,n-1\}},$$
$$\text{where} \quad \psi_{\mathbf{p},\mathbf{q}} = \sum_{\substack{\mathbf{a}\subseteq\mathbf{p},\mathbf{b}\subseteq\mathbf{q}\\ \mathbf{c}=\mathbf{p}\setminus\mathbf{a}\\ \mathbf{d}=\mathbf{q}\setminus\mathbf{b}}} \phi_{\mathbf{a},\mathbf{b}} \cdot \varphi_{\mathbf{c},\mathbf{d}}. \tag{4}$$

For each $\mathbf{p}$, $\mathbf{q}$, Proposition 2.7(2), Remark 3.3 and (3) imply the following R-module isomorphism

$$B_{\mathbf{p},\mathbf{q}}^{(k)} \cong \begin{cases} B_{\{k-p_r,\ldots,k-p_1,q_1+1,\ldots,q_s+1\}} \oplus B_{\{k-p_r,\ldots,k-p_1,k,q_1+1,\ldots,q_s+1\}}, \\ \text{or} \\ B_{\{1,k-p_r,\ldots,k-p_1,q_2+1,\ldots,q_s+1\}} \oplus B_{\{1,k-p_r,\ldots,k-p_1,k,q_2+1,\ldots,q_s+1\}}. \end{cases} \tag{5}$$

Therefore one gets an R-module isomorphism $Q_k \cong \bigoplus_{\mathbf{i}\subseteq[n]} B_{\mathbf{i}}$. Set $Q = Q_k$.

Assume that $\mathbf{p}, \mathbf{p}' \subseteq \{1, \ldots, k-2\}$ and $\mathbf{q}, \mathbf{q}' \subseteq \{k-1, \ldots, n-1\}$ are such that $\mathbf{p}\cap\mathbf{p}' = \emptyset$ and $\mathbf{q}\cap\mathbf{q}' = \emptyset$. By Proposition 2.7(5) and Remark 3.3, the R_k-module $B_{\mathbf{p},\mathbf{q}}^{(k)} \otimes_{R_k} B_{\mathbf{p}',\mathbf{q}'}^{(k)}$ is a semidualizing and so $B_{\mathbf{p},\mathbf{q}}^{(k)} \otimes_{R_k} B_{\mathbf{p}',\mathbf{q}'}^{(k)} = B_{\mathbf{p}\cup\mathbf{p}',\mathbf{q}\cup\mathbf{q}'}^{(k)}$. If $\phi_{\mathbf{p},\mathbf{q}} \in B_{\mathbf{p},\mathbf{q}}^{(k)}$ and $\varphi_{\mathbf{p}',\mathbf{q}'} \in B_{\mathbf{p}',\mathbf{q}'}^{(k)}$, then by the isomorphism (5), one has $\phi_{\mathbf{p},\mathbf{q}} = (\beta_{\mathbf{p},\mathbf{q}}, \gamma_{\mathbf{p},\mathbf{q}})$ and $\varphi_{\mathbf{p}',\mathbf{q}'} = (\beta_{\mathbf{p}',\mathbf{q}'}, \gamma_{\mathbf{p}',\mathbf{q}'})$, so that

$$\phi_{\mathbf{p},\mathbf{q}} \cdot \varphi_{\mathbf{p}',\mathbf{q}'} = (\beta_{\mathbf{p},\mathbf{q}} \cdot \beta_{\mathbf{p}',\mathbf{q}'},\ \beta_{\mathbf{p},\mathbf{q}} \cdot \gamma_{\mathbf{p}',\mathbf{q}'} + \beta_{\mathbf{p}',\mathbf{q}'} \cdot \gamma_{\mathbf{p},\mathbf{q}}).$$

Thus by means of the ring structure on Q_k, (4), one can see that the resulting ring structure on Q is as claimed.

The next step is to introduce the ideals $I_1, \dots, I_n$. We set

$$I_\ell = (\underbrace{0 \oplus \cdots \oplus 0}_{2^{n-1}}) \oplus \Big(\bigoplus_{\mathbf{i} \subseteq [n], \ell \in \mathbf{i}} B_{\mathbf{i}} \Big), \qquad 1 \leqslant \ell \leqslant n,$$

which is an ideal of Q. Also we have the following sequence of R-isomorphisms which preserve ring isomorphisms:

$$\begin{aligned} Q/(I_1 + \cdots + I_n) &= \Big(\bigoplus_{\mathbf{i} \subseteq [n]} B_{\mathbf{i}} \Big) \Big/ \Big(\sum_{\ell=1}^{n} (\underbrace{0 \oplus \cdots \oplus 0}_{2^{n-1}}) \oplus \Big(\bigoplus_{\mathbf{i} \subseteq [n],\, \ell \in \mathbf{i}} B_{\mathbf{i}} \Big) \Big) \\ &\cong \Big(\bigoplus_{\mathbf{i} \subseteq [n]} B_{\mathbf{i}} \Big) \Big/ \Big(\bigoplus_{\mathbf{i} \subseteq [n], \mathbf{i} \neq \emptyset} B_{\mathbf{i}} \Big) \\ &\cong R. \end{aligned}$$

Note that each ideal $I_{k,\ell}$, $1 \leqslant \ell \leqslant n-1$, of Q_k has the form $I_{k,\ell} = (\underbrace{0 \oplus \cdots \oplus 0}_{2^{n-2}}) \oplus \big(\bigoplus_{\ell \in \mathbf{p} \cup \mathbf{q}} B^{(k)}_{\mathbf{p},\mathbf{q}} \big)$. Then, by (5), one has the following R-module isomorphism

$$I_{k,\ell} \cong \begin{cases} I_{k-\ell}, & \text{if } 1 \leqslant \ell \leqslant k-1, \\ I_{\ell+1}, & \text{if } k \leqslant \ell \leqslant n-1. \end{cases}$$

Also, by means of the ring isomorphism $Q_k \to Q$, we have the natural correspondence between ideals:

$$I_{k,\ell} \xleftrightarrow{\text{correspond}} \begin{cases} I_{k-\ell}, & \text{if } 1 \leqslant \ell \leqslant k-1, \\ I_{\ell+1}, & \text{if } k \leqslant \ell \leqslant n-1. \end{cases}$$

Therefore for each $\Lambda \subseteq [n] \setminus \{k\}$, there is a ring isomorphism $Q/\big(\sum_{\ell \in \Lambda} I_\ell\big) \cong Q_k/\big(\sum_{\ell \in \Lambda'} I_{k,\ell}\big)$, for some $\Lambda' \subseteq [n-1]$.

The proof of Theorem 3.9, which is inspired by the proof of [11, Theorem 3.2], is rather technical and needs some preparatory lemmas.

LEMMA 3.11. *Assume that* $\Lambda \subseteq [n]$. *Under the hypothesis of Theorem 3.9, if* $[n] \setminus \Lambda = \{b_1, \dots, b_t\}$ *with* $1 \leqslant b_1 < \cdots < b_t \leqslant n$, *then there is an* R*-isomorphism*

$$R_\Lambda \cong \bigoplus_{\mathbf{i} \subseteq \{b_1, \dots, b_t\}} B_{\mathbf{i}}$$

which induces a ring structure on R_Λ *as follows: for elements* $(\alpha_{\mathbf{i}})_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}}$ *and* $(\theta_{\mathbf{i}})_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}}$ *of* R_Λ,

$$(\alpha_{\mathbf{i}})_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}}(\theta_{\mathbf{i}})_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}} = (\sigma_{\mathbf{i}})_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}}, \quad \textit{where } \sigma_{\mathbf{i}} = \sum_{\substack{\mathbf{v}\subseteq\mathbf{i}\\ \mathbf{w}=\mathbf{i}\setminus\mathbf{v}}} \alpha_{\mathbf{v}}\cdot\theta_{\mathbf{w}}.$$

PROOF. We prove by induction on n. The case $n = 1$ is clear. The case $n = 2$ is proved in [11]. Assume that $n > 2$ and the assertion holds true for $n - 1$.

If $\Lambda = [n]$, there is nothing to prove. Suppose that $|\Lambda| \leqslant n - 1$ then there exists $k \in [n]$ such that $\Lambda \subseteq [n] \setminus \{k\}$. Thus, by Construction 3.10, there exists a subset Λ' of $[n - 1]$ such that $R_\Lambda \cong Q_k/(\sum_{\ell\in\Lambda'} I_{k,\ell})$ as ring isomorphism.

Note that $|[n-1]\setminus\Lambda'| = t-1$. Set $[n-1]\setminus\Lambda' = \{d_1, \dots, d_u, d_{u+1}, \dots, d_{t-1}\}$ such that $1 \leqslant d_1 < \cdots < d_u < k-1$ and $k-1 \leqslant d_{u+1} < \cdots < d_{t-1} \leqslant n-1$. Then by induction there exists an R_k-isomorphism

$$Q_k/\Big(\sum_{\ell\in\Lambda'} I_{k,\ell}\Big) \cong \bigoplus_{\substack{\mathbf{p}\subseteq\{d_1,\dots,d_u\}\\ \mathbf{q}\subseteq\{d_{u+1},\dots,d_{t-1}\}}} B^{(k)}_{\mathbf{p},\mathbf{q}}.$$

Proceeding as Construction 3.10, there is an R-isomorphism

$$\Big(\bigoplus_{\substack{\mathbf{p}\subseteq\{d_1,\dots,d_u\}\\ \mathbf{q}\subseteq\{d_{u+1},\dots,d_{t-1}\}}} B^{(k)}_{\mathbf{p},\mathbf{q}}\Big) \cong \Big(\bigoplus_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}} B_{\mathbf{i}}\Big).$$

Therefore one has an R-isomorphism $R_\Lambda \cong \bigoplus_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}} B_{\mathbf{i}}$. Similar to Construction 3.10, R_Λ has the desired ring structure.

LEMMA 3.12. *Under the hypothesis of Theorem 3.9, if* $\Gamma \subsetneq \Lambda \subseteq [n]$, *we have* $\operatorname{Ext}^{\geqslant 1}_{R_\Gamma}(R_\Lambda, R_\Gamma) = 0$ *and* $\operatorname{Hom}_{R_\Gamma}(R_\Lambda, R_\Gamma)$ *is a non-free semidualizing* R_Λ*-module.*

PROOF. The case $n = 1$ is clear and the case $n = 2$ is proved in [11, Lemma 3.8]. Let $n > 2$ and suppose that the assertion is settled for $n - 1$.

First assume that $\Lambda = [n]$. Set $[n] \setminus \Gamma = \{a_1, \dots, a_s\}$ with $1 \leqslant a_1 < \cdots < a_s \leqslant n$. By Lemma 3.11, $R_\Gamma \cong \bigoplus_{\mathbf{i}\subseteq\{a_1,\dots,a_s\}} B_{\mathbf{i}}$. By Proposition 2.7(4) and Remark 3.3, $[B_{\{a_1,\dots,a_s\}}] \trianglelefteq [B_{\mathbf{i}}]$ and $\operatorname{Hom}_R(B_{\mathbf{i}}, B_{\{a_1,\dots,a_s\}}) \cong B_{\{a_1,\dots,a_s\}\setminus\mathbf{i}}$, for all

$\mathbf{i} \subseteq \{a_1, \ldots, a_s\}$. Therefore there are R-isomorphisms

$$\begin{aligned}\operatorname{Hom}_R(R_\Gamma, B_{\{a_1,\ldots,a_s\}}) &\cong \operatorname{Hom}_R\Bigg(\bigoplus_{\mathbf{i}\subseteq\{a_1,\ldots,a_s\}} B_{\mathbf{i}}, B_{\{a_1,\ldots,a_s\}}\Bigg)\\ &\cong \bigoplus_{\mathbf{i}\subseteq\{a_1,\ldots,a_s\}} B_{\mathbf{i}} \cong R_\Gamma\end{aligned}$$

and, for all $i \geqslant 1$,

$$\operatorname{Ext}^i_R(R_\Gamma, B_{\{a_1,\ldots,a_s\}}) \cong \operatorname{Ext}^i_R\Bigg(\bigoplus_{\mathbf{i}\subseteq\{a_1,\ldots,a_s\}} B_{\mathbf{i}}, B_{\{a_1,\ldots,a_s\}}\Bigg) = 0.$$

Let $\mathbf{E}$ be an injective resolution of $B_{\{a_1,\ldots,a_s\}}$ as an R-module. Thus $\operatorname{Hom}_R(R_\Gamma, \mathbf{E})$ is an injective resolution of R_Γ as an R_Γ-module. Note that the composition of natural homomorphisms $R \to R_\Gamma \to R$ is the identity id_R. Therefore

$$\operatorname{Hom}_{R_\Gamma}(R, \operatorname{Hom}_R(R_\Gamma, \mathbf{E})) \cong \operatorname{Hom}_R(R \otimes_{R_\Gamma} R_\Gamma, \mathbf{E}) \cong \operatorname{Hom}_R(R, \mathbf{E}) \cong \mathbf{E}\,.$$

Hence

$$\begin{aligned}\operatorname{Ext}^i_{R_\Gamma}(R, R_\Gamma) &\cong \mathrm{H}^i(\operatorname{Hom}_{R_\Gamma}(R, \operatorname{Hom}_R(R_\Gamma, \mathbf{E})))\\ &\cong \mathrm{H}^i(\mathbf{E})\\ &\cong \begin{cases} 0, & \text{if } i > 0,\\ B_{\{a_1,\ldots,a_s\}}, & \text{if } i = 0.\end{cases}\end{aligned}$$

As $\{a_1, \ldots, a_s\} \neq \emptyset$, the R-module $B_{\{a_1,\ldots,a_s\}}$ is a non-free semidualizing.

Now assume that $|\Lambda| \leqslant n-1$. There exist $k \in [n]$, and subsets Γ', Λ' of $[n-1]$ such that there are R-isomorphisms and ring isomorphisms $R_\Gamma \cong Q_k/\big(\sum_{\ell\in\Gamma'} I_{k,\ell}\big)$ and $R_\Lambda \cong Q_k/\big(\sum_{\ell\in\Lambda'} I_{k,\ell}\big)$, where Q_k and $I_{k,\ell}$ are as in Construction 3.10. By induction we have

$$\operatorname{Ext}^i_{R_\Gamma}(R_\Lambda, R_\Gamma) \cong \operatorname{Ext}^i_{Q_k/(\sum_{\ell\in\Gamma'} I_{k,\ell})}\Bigg(Q_k\Big/\Bigg(\sum_{\ell\in\Lambda'} I_{k,\ell}\Bigg), Q_k\Big/\Bigg(\sum_{\ell\in\Gamma'} I_{k,\ell}\Bigg)\Bigg) = 0$$

for all $i \geqslant 1$, and

$$\operatorname{Hom}_{R_\Gamma}(R_\Lambda, R_\Gamma) \cong \operatorname{Hom}_{Q_k/(\sum_{\ell\in\Gamma'} I_{k,\ell})}\Bigg(Q_k\Big/\Bigg(\sum_{\ell\in\Lambda'} I_{k,\ell}\Bigg), Q_k\Big/\Bigg(\sum_{\ell\in\Gamma'} I_{k,\ell}\Bigg)\Bigg)$$

is a non-free semidualizing $Q_k/\big(\sum_{\ell\in\Lambda'} I_{k,\ell}\big)$-module. Then $\operatorname{Hom}_{R_\Gamma}(R_\Lambda, R_\Gamma)$ is a non-free semidualizing R_Λ-module.

LEMMA 3.13. *Under the hypothesis of Theorem 3.9, if* Λ *and* Γ *are two subsets of* $[n]$*, then* $\operatorname{Tor}^{R_{\Lambda\cup\Gamma}}_{\geqslant 1}(R_\Lambda, R_\Gamma) = 0$*. Moreover, there is an* R_Λ*-algebra isomorphism* $R_\Lambda \otimes_{R_{\Lambda\cup\Gamma}} R_\Gamma \cong R_{\Lambda\cap\Gamma}$*.*

PROOF. We prove by induction. If $n = 1$, there is nothing to prove. The case $n = 2$ is proved in [11, Lemma 3.9]. Let $n > 2$ and suppose that the assertion holds true for $n-1$. First assume that $\Lambda\cup\Gamma = [n]$ and set $[n]\setminus\Lambda = \{b_1, \dots, b_t\}$, $[n] \setminus \Gamma = \{a_1, \dots, a_s\}$. Then $[n] \setminus (\Lambda \cap \Gamma) = \{b_1, \dots, b_t, a_1, \dots, a_s\}$. By Lemma 3.11, $R_\Lambda \cong \bigoplus_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}} B_{\mathbf{i}}$ and $R_\Gamma \cong \bigoplus_{\mathbf{u}\subseteq\{a_1,\dots,a_s\}} B_{\mathbf{u}}$.

As $\{b_1, \dots, b_t\} \cap \{a_1, \dots, a_s\} = \emptyset$, for each $\mathbf{i} \subseteq \{b_1, \dots, b_t\}$ and $\mathbf{u} \subseteq \{a_1, \dots, a_s\}$, by Proposition 2.7(5) and Remark 3.3, one has $B_{\mathbf{i}} \in \mathscr{A}_{B_{\mathbf{u}}}(R)$ and so $\operatorname{Tor}^R_{\geqslant 1}(B_{\mathbf{i}}, B_{\mathbf{u}}) = 0$. Hence $\operatorname{Tor}^R_{\geqslant 1}(R_\Lambda, R_\Gamma) = 0$.

By Proposition 2.7(5) and Remark 3.3, the R-module $B_{\mathbf{i}} \otimes_R B_{\mathbf{u}}$ is semidualizing and so $B_{\mathbf{i}} \otimes_R B_{\mathbf{u}} = B_{\mathbf{i}\cup\mathbf{u}}$. Therefore one has the natural R-module isomorphism

$$\eta\colon R_\Lambda \otimes_R R_\Gamma \longrightarrow R_{\Lambda\cap\Gamma},$$
$$\eta\big((\alpha_{\mathbf{i}})_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}} \otimes (\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\dots,a_s\}}\big) = (\alpha_{\mathbf{i}} \cdot \theta_{\mathbf{u}})_{\substack{\mathbf{i}\subseteq\{b_1,\dots,b_t\}\\ \mathbf{u}\subseteq\{a_1,\dots,a_s\}}}$$

It is routine to check that η is also a ring isomorphism.

On the other hand the natural maps

$$\zeta\colon R_\Lambda \to R_\Lambda \otimes_R R_\Gamma, \quad \zeta\big((\alpha_{\mathbf{i}})_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}}\big) = (\alpha_{\mathbf{i}})_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}} \otimes (\dot{\theta}_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\dots,a_s\}}$$

and

$$\varepsilon\colon R_\Lambda \to R_{\Lambda\cap\Gamma}, \quad \varepsilon\big((\alpha_{\mathbf{i}})_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}}\big) = (\chi_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\dots,a_s,b_1,\dots,b_t\}},$$

where

$$\dot{\theta}_{\mathbf{u}} = \begin{cases} 0, & \text{if } \mathbf{u} \neq \emptyset, \\ 1, & \text{if } \mathbf{u} = \emptyset, \end{cases} \qquad \text{and} \qquad \chi_{\mathbf{v}} = \begin{cases} \alpha_{\mathbf{v}}, & \text{if } \mathbf{v} \cap \{a_1, \dots, a_s\} = \emptyset, \\ 0, & \text{if } \mathbf{v} \cap \{a_1, \dots, a_s\} \neq \emptyset, \end{cases}$$

are ring homomorphisms. It is easy to check that $\eta\zeta = \varepsilon$. Hence $R_\Lambda \otimes_R R_\Gamma \xrightarrow{\eta} R_{\Lambda\cap\Gamma}$ is an R_Λ-algebra isomorphism.

Now let $\Lambda \cup \Gamma \subsetneq [n]$, then, by Construction 3.10, there exist $k \in [n]$ and $\Lambda', \Gamma' \subseteq [n-1]$ such that there are R-isomorphisms and ring isomorphisms

$$R_\Lambda \cong Q_k\Big/\Big(\sum_{\ell\in\Lambda'} I_{k,\ell}\Big), \quad R_\Gamma \cong Q_k\Big/\Big(\sum_{\ell\in\Gamma'} I_{k,\ell}\Big),$$
$$R_{\Lambda\cup\Gamma} \cong Q_k\Big/\Big(\sum_{\ell\in\Lambda'\cup\Gamma'} I_{k,\ell}\Big) \quad \text{and} \quad R_{\Lambda\cap\Gamma} \cong Q_k\Big/\Big(\sum_{\ell\in\Lambda'\cap\Gamma'} I_{k,\ell}\Big).$$

Thus, by induction, for all $i \geqslant 1$

$$\begin{aligned}\operatorname{Tor}_i^{R_{\Lambda\cup\Gamma}}(R_\Lambda, R_\Gamma) &\cong \operatorname{Tor}_i^{Q_k/(\sum_{\ell\in\Lambda'\cup\Gamma'} I_{k,\ell})}\Bigl(Q_k\Big/\Bigl(\sum_{\ell\in\Lambda'} I_{k,\ell}\Bigr), Q_k\Big/\Bigl(\sum_{\ell\in\Gamma'} I_{k,\ell}\Bigr)\Bigr)\\ &= 0\end{aligned}$$

and there is a $Q_k/\bigl(\sum_{\ell\in\Lambda'} I_{k,\ell}\bigr)$-algebra isomorphism, and so R_Λ-algebra isomorphism, as follows:

$$\begin{aligned}R_\Lambda \otimes_{R_{\Lambda\cup\Gamma}} R_\Gamma &\cong Q_k\Big/\Bigl(\sum_{\ell\in\Lambda'} I_{k,\ell}\Bigr) \otimes_{Q_k/(\sum_{\ell\in\Lambda'\cup\Gamma'} I_{k,\ell})} Q_k\Big/\Bigl(\sum_{\ell\in\Gamma'} I_{k,\ell}\Bigr)\\ &\cong Q_k\Big/\Bigl(\sum_{\ell\in\Lambda'\cap\Gamma'} I_{k,\ell}\Bigr)\\ &\cong R_{\Lambda\cap\Gamma}.\end{aligned}$$

LEMMA 3.14. *Under the hypothesis of Theorem 3.9, if* Λ *and* Γ *are two subsets of* $[n]$*, then* $\operatorname{Tor}_{\geqslant 1}^{R_\Lambda}(R_{\Lambda\cup\Gamma}, R_{\Lambda\cap\Gamma}) = 0$*. Moreover, there is an* $R_{\Lambda\cap\Gamma}$*-module isomorphism* $R_{\Lambda\cup\Gamma} \otimes_{R_\Lambda} R_{\Lambda\cap\Gamma} \cong R_\Gamma$.

PROOF. It is proved by induction on n. If $n = 1$, there is nothing to prove. The case $n = 2$ is proved in [11, Lemma 3.11]. Let $n > 2$ and suppose that the assertion holds true for $n - 1$.

First assume that $\Lambda \cup \Gamma = [n]$. Let $\mathbf{P}$ be an R-projective resolution of R_Γ. Lemma 3.13 implies that $R_\Lambda \otimes_R \mathbf{P}$ is an R_Λ-projective resolution of $R_\Lambda \otimes_R R_\Gamma \cong R_{\Lambda\cap\Gamma}$. One has the following natural isomorphisms

$$R \otimes_{R_\Lambda} (R_\Lambda \otimes_R \mathbf{P}) \cong (R \otimes_{R_\Lambda} R_\Lambda) \otimes_R \mathbf{P} \cong R \otimes_R \mathbf{P} \cong \mathbf{P}$$

and then, for all $i \geqslant 1$,

$$\operatorname{Tor}_i^{R_\Lambda}(R, R_{\Lambda\cap\Gamma}) \cong \mathrm{H}_i(R \otimes_{R_\Lambda} (R_\Lambda \otimes_R \mathbf{P})) \cong \mathrm{H}_i(\mathbf{P}) = 0.$$

Set $[n] \setminus \Lambda = \{b_1, \dots, b_t\}$ and $[n] \setminus \Gamma = \{a_1, \dots, a_s\}$. Then $[n] \setminus (\Lambda \cap \Gamma) = \{b_1, \dots, b_t, a_1, \dots, a_s\}$. Consider the R-module isomorphism $\xi\colon R_\Gamma \xrightarrow{\cong} R \otimes_{R_\Lambda} R_{\Lambda\cap\Gamma}$ which is the composition

$$R_\Gamma \xrightarrow{\cong} R \otimes_R R_\Gamma \xrightarrow{\cong} R \otimes_{R_\Lambda} (R_\Lambda \otimes_R R_\Gamma) \xrightarrow[R\otimes\eta]{\cong} R \otimes_{R_\Lambda} R_{\Lambda\cap\Gamma}$$

given by

$$\begin{aligned}(\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\dots,a_s\}} \mapsto 1 \otimes (\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\dots,a_s\}} &\mapsto 1 \otimes [(\dot{\alpha}_{\mathbf{i}})_{\mathbf{i}\subseteq\{b_1,\dots,b_t\}} \otimes (\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\dots,a_s\}}]\\ &\mapsto 1 \otimes (\lambda_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\dots,a_s,b_1,\dots,b_t\}},\end{aligned}$$

where

$$\dot{\alpha}_{\mathbf{i}} = \begin{cases} 0, & \text{if } \mathbf{i} \neq \emptyset, \\ 1, & \text{if } \mathbf{i} = \emptyset, \end{cases} \qquad \text{and} \qquad \lambda_{\mathbf{v}} = \begin{cases} \theta_{\mathbf{v}}, & \text{if } \mathbf{v} \cap \{b_1, \ldots, b_t\} = \emptyset, \\ 0, & \text{if } \mathbf{v} \cap \{b_1, \ldots, b_t\} \neq \emptyset. \end{cases}$$

We claim that ξ is an $R_{\Lambda\cap\Gamma}$-module isomorphism.

Proof of the claim. The $R_{\Lambda\cap\Gamma}$-module structure of R_Γ, which is given via the natural surjection $R_{\Lambda\cap\Gamma} \to R_\Gamma$, is described as

$$(\gamma_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}(\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}} = (\gamma_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}}(\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}},$$

where $(\gamma_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}$ is an element of $R_{\Lambda\cap\Gamma}$. In the following we check that

$$\xi\big((\gamma_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}(\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}}\big) \\ = (\gamma_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}\big[\xi\big((\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}}\big)\big].$$

Note that

$$\begin{aligned} \xi\big((\gamma_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}(\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}}\big) &= \xi\big((\gamma_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}}(\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}}\big) \\ &= \xi\big((\sigma_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}}\big) \\ &= 1 \otimes (\mu_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}, \end{aligned}$$

where $(\sigma_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}} = (\gamma_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}}(\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}}$ and

$$\mu_{\mathbf{v}} = \begin{cases} \sigma_{\mathbf{v}}, & \text{if } \mathbf{v} \cap \{b_1, \ldots, b_t\} = \emptyset, \\ 0 & \text{if } \mathbf{v} \cap \{b_1, \ldots, b_t\} \neq \emptyset. \end{cases}$$

On the other hand

$$\begin{aligned} &(\gamma_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}[\xi((\theta_{\mathbf{u}})_{\mathbf{u}\subseteq\{a_1,\ldots,a_s\}})] \\ &\quad= (\gamma_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}[1 \otimes (\lambda_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}] \\ &\quad= 1 \otimes [(\gamma_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}(\lambda_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}] \\ &\quad= 1 \otimes (\varrho_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}} \\ &\quad= [1 \otimes (\mu_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}] + [1 \otimes \delta], \end{aligned}$$

where $\delta = (\delta_{\mathbf{v}})_{\mathbf{v}\subseteq\{a_1,\ldots,a_s,b_1,\ldots,b_t\}}$ with

$$\delta_{\mathbf{v}} = \begin{cases} 0, & \text{if } \mathbf{v} \cap \{b_1, \ldots, b_t\} = \emptyset, \\ \varrho_{\mathbf{v}}, & \text{if } \mathbf{v} \cap \{b_1, \ldots, b_t\} \neq \emptyset. \end{cases}$$

It is enough to show that $1 \otimes \delta = 0$. To this end, we have

$$1 \otimes \delta = \sum_{\substack{\mathbf{w} \subseteq \{a_1,\dots,a_s,b_1,\dots,b_t\} \\ \mathbf{w} \cap \{b_1,\dots,b_t\} \neq \emptyset}} 1 \otimes \delta(\mathbf{w}),$$

where $\delta(\mathbf{w}) = (\delta(\mathbf{w})_{\mathbf{v}})_{\mathbf{v} \subseteq \{a_1,\dots,a_s,b_1,\dots,b_t\}}$ with

$$\delta(\mathbf{w})_{\mathbf{v}} = \begin{cases} 0, & \text{if } \mathbf{v} \neq \mathbf{w}, \\ \delta_{\mathbf{w}}, & \text{if } \mathbf{v} = \mathbf{w}. \end{cases}$$

For each $\mathbf{w}$, there exist $\mathbf{w}' \subseteq \{b_1, \dots, b_t\}$ and $\mathbf{w}'' \subseteq \{a_1, \dots, a_s\}$ with $\mathbf{w}' \cup \mathbf{w}'' = \mathbf{w}$. Thus $B_{\mathbf{w}'} \otimes_R B_{\mathbf{w}''} \overset{\rho_{\mathbf{w}}}{\cong} B_{\mathbf{w}}$ and there exist $\delta'_{\mathbf{w}} \in B_{\mathbf{w}'}$ and $\delta''_{\mathbf{w}} \in B_{\mathbf{w}''}$ such that $\delta_{\mathbf{w}} = \rho_{\mathbf{w}}(\delta'_{\mathbf{w}} \otimes \delta''_{\mathbf{w}})$.

Set $\alpha(\mathbf{w}) = \big(\alpha(\mathbf{w})_{\mathbf{i}}\big)_{\mathbf{i} \subseteq \{b_1,\dots,b_t\}}$, where

$$\alpha(\mathbf{w})_{\mathbf{i}} = \begin{cases} 0, & \text{if } \mathbf{i} \neq \mathbf{w}', \\ \delta'_{\mathbf{w}}, & \text{if } \mathbf{i} = \mathbf{w}'. \end{cases}$$

As the R_Λ-module structure on R is given via the natural surjection $R_\Lambda \longrightarrow R$, and $\alpha(\mathbf{w})$ is an element of the kernel of this map, $0 \oplus \big(\bigoplus_{\mathbf{i} \subseteq \{b_1,\dots,b_t\}, \mathbf{i} \neq \emptyset} B_{\mathbf{i}}\big)$, we have $1\alpha(\mathbf{w}) = 0$. Set $\beta(\mathbf{w}) = (\beta(\mathbf{w})_{\mathbf{v}})_{\mathbf{v} \subseteq \{a_1,\dots,a_s,b_1,\dots,b_t\}}$, where

$$\beta(\mathbf{w})_{\mathbf{v}} = \begin{cases} 0, & \text{if } \mathbf{v} \neq \mathbf{w}'', \\ \delta''_{\mathbf{w}}, & \text{if } \mathbf{v} = \mathbf{w}''. \end{cases}$$

Note that $\beta(\mathbf{w})$ is an element of $R_{\Lambda \cap \Gamma}$ and $\delta(\mathbf{w}) = \alpha(\mathbf{w})\beta(\mathbf{w})$. Then

$$\begin{aligned} 1 \otimes \delta &= \sum_{\mathbf{w}} 1 \otimes \delta(\mathbf{w}) = \sum_{\mathbf{w}} 1 \otimes [\alpha(\mathbf{w})\beta(\mathbf{w})] \\ &= \sum_{\mathbf{w}} [1\alpha(\mathbf{w})] \otimes \beta(\mathbf{w}) = \sum_{\mathbf{w}} 0 \otimes \beta(\mathbf{w}) = 0. \end{aligned}$$

Therefore the claim is proved and also the assertion holds in the case $\Lambda \cup \Gamma = [n]$.

We treat the case $\Lambda \cup \Gamma \subsetneq [n]$ by induction and its details are similar to the proof of Lemma 3.13.

PROOF OF THEOREM 3.9. (1) is proved in Construction 3.10.

(2) is proved by induction on n. The case $n = 1$ is clear from the assumptions. Let $n > 1$ and suppose the claim is settled for $n - 1$. If $\Lambda = [n]$, then $R_\Lambda \cong R$ and is Cohen-Macaulay with the dualizing module D and is

not Gorenstein. Let $\Lambda \subsetneq [n]$. There exists $k \in [n]$ such that $\Lambda \subseteq [n] \setminus \{k\}$. By Construction 3.10, there exists a subset $\Lambda' \neq \emptyset$ of $[n-1]$ such that $R_\Lambda \cong Q_k/(\sum_{\ell \in \Lambda'} I_{k,\ell})$ as ring isomorphism. Thus, by induction, R_Λ is non-Gorenstein Cohen-Macaulay ring with dualizing module.

(3). It is clear that $\prod_{\ell \in \Lambda} I_\ell \subseteq \bigcap_{\ell \in \Lambda} I_\ell$. Let $\alpha = (\alpha_{\mathbf{i}})_{\mathbf{i} \subseteq [n]}$ be an element of $\bigcap_{\ell \in \Lambda} I_\ell$. Then, by Construction 3.10, $\alpha_{\mathbf{i}} = 0$ for all $\mathbf{i} \subseteq [n]$ with $\Lambda \nsubseteq \mathbf{i}$. We have $\alpha = \sum_{\Lambda \subseteq \mathbf{v} \subseteq [n]} \alpha(\mathbf{v})$, where $\alpha(\mathbf{v}) = (\alpha(\mathbf{v})_{\mathbf{i}})_{\mathbf{i} \subseteq [n]}$ with

$$\alpha(\mathbf{v})_{\mathbf{i}} = \begin{cases} 0, & \text{if } \mathbf{i} \neq \mathbf{v}, \\ \alpha_{\mathbf{v}} & \text{if } \mathbf{i} = \mathbf{v}. \end{cases}$$

Set $\Lambda = \{a_1, \ldots, a_m\}$. If $\mathbf{v} \subseteq [n]$ is such that $\Lambda \subseteq \mathbf{v}$, then $\mathbf{v} = \{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_{m-1}\} \cup (\mathbf{v} \setminus \{a_1, \ldots, a_{m-1}\})$. Thus

$$B_{\mathbf{v}} \overset{\Phi}{\cong} B_{\{a_1\}} \otimes_R \cdots \otimes_R B_{\{a_{m-1}\}} \otimes_R B_{\mathbf{v} \setminus \{a_1, \ldots, a_{m-1}\}}.$$

Therefore there exist $\theta_{\mathbf{v},m} \in B_{\mathbf{v} \setminus \{a_1, \ldots, a_{m-1}\}}$ and $\theta_{\mathbf{v},r} \in B_{\{a_r\}}$, $1 \leqslant r < m$, such that $\alpha_{\mathbf{v}} = \Phi(\theta_{\mathbf{v},1} \otimes \cdots \otimes \theta_{\mathbf{v},m-1} \otimes \theta_{\mathbf{v},m})$. Set $\varphi(\mathbf{v}, r) = (\varphi(\mathbf{v}, r)_{\mathbf{i}})_{\mathbf{i} \subseteq [n]}$, $1 \leqslant r \leqslant m$, where, for $1 \leqslant r < m$,

$$\varphi(\mathbf{v}, r)_{\mathbf{i}} = \begin{cases} 0, & \text{if } \mathbf{i} \neq \{a_r\}, \\ \theta_{\mathbf{v},r}, & \text{if } \mathbf{i} = \{a_r\} \end{cases}$$

and

$$\varphi(\mathbf{v}, m)_{\mathbf{i}} = \begin{cases} 0, & \text{if } \mathbf{i} \neq \mathbf{v} \setminus \{a_1, \ldots, a_{m-1}\}, \\ \theta_{\mathbf{v},m}, & \text{if } \mathbf{i} = \mathbf{v} \setminus \{a_1, \ldots, a_{m-1}\}. \end{cases}$$

Note that $\varphi(\mathbf{v}, r) \in I_{a_r}$, $1 \leqslant r \leqslant m$. Hence $\varphi(\mathbf{v}, 1) \ldots \varphi(\mathbf{v}, m-1)\varphi(\mathbf{v}, m) \in \prod_{\ell \in \Lambda} I_\ell$. On the other hand $\varphi(\mathbf{v}, 1) \ldots \varphi(\mathbf{v}, m-1)\varphi(\mathbf{v}, m) = \alpha(\mathbf{v})$. Thus $\alpha(\mathbf{v})$ is an element of $\prod_{\ell \in \Lambda} I_\ell$ and so $\alpha \in \prod_{\ell \in \Lambda} I_\ell$.

(4) follows from by Remark 2.2 and Lemma 3.12.

(5). Let $\mathbf{P}$ be a projective resolution of $R_{\Lambda \cup \Gamma}$ over R_Λ. Lemma 3.14 implies that the complex $\mathbf{P} \otimes_{R_\Lambda} R_{\Lambda \cap \Gamma}$ is a $R_{\Lambda \cap \Gamma}$-projective resolution of $R_{\Lambda \cup \Gamma} \otimes_{R_\Lambda} R_{\Lambda \cap \Gamma} \cong R_\Gamma$. From the isomorphisms

$$(\mathbf{P} \otimes_{R_\Lambda} R_{\Lambda \cap \Gamma}) \otimes_{R_{\Lambda \cap \Gamma}} R_\Lambda \cong \mathbf{P} \otimes_{R_\Lambda} R_\Lambda \cong \mathbf{P}$$

one gets

$$\mathrm{Tor}_i^{R_{\Lambda \cap \Gamma}}(R_\Gamma, R_\Lambda) \cong \mathrm{H}_i((\mathbf{P} \otimes_{R_\Lambda} R_{\Lambda \cap \Gamma}) \otimes_{R_{\Lambda \cap \Gamma}} R_\Lambda) \cong \mathrm{H}_i(\mathbf{P}) = 0.$$

for all $i \geqslant 1$. There is a natural isomorphism $R_\Lambda \otimes_{R_{\Lambda \cap \Gamma}} R_\Gamma \cong R_{\Lambda \cup \Gamma}$ which is both an $R_{\Lambda \cap \Gamma}$- and an R_Γ-isomorphism.

Let $\mathbf{P}'$ be an $R_{\Lambda\cap\Gamma}$-projective resolution of R_Λ. As seen in the above, $\mathbf{P}'\otimes_{R_{\Lambda\cap\Gamma}} R_\Gamma$ is a projective resolution of $R_{\Lambda\cup\Gamma}$ over R_Γ. Therefore we have

$$\begin{aligned}\operatorname{Ext}^i_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) &\cong \mathrm{H}^i(\operatorname{Hom}_{R_{\Lambda\cap\Gamma}}(\mathbf{P}', R_\Gamma))\\ &\cong \mathrm{H}^i(\operatorname{Hom}_{R_\Gamma}(\mathbf{P}'\otimes_{R_{\Lambda\cap\Gamma}} R_\Gamma, R_\Gamma))\\ &\cong \operatorname{Ext}^i_{R_\Gamma}(R_{\Lambda\cup\Gamma}, R_\Gamma),\end{aligned}$$

for all $i \geqslant 0$. By (4), $\operatorname{G-dim}_{R_\Gamma} R_{\Lambda\cup\Gamma} = 0$, and so one gets $\operatorname{Ext}^{\geqslant 1}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) = 0$. Also, by (4), $\operatorname{Hom}_{R_\Gamma}(R_{\Lambda\cup\Gamma}, R_\Gamma)$ is a non-free semidualizing $R_{\Lambda\cup\Gamma}$-module and thus $\operatorname{Hom}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma)$ is not cyclic.

(6). As $R_{\Lambda\cap\Gamma} = Q/\left(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\right)$ and

$$R_\Lambda = Q\Big/\left(\sum_{\ell\in\Lambda} I_\ell\right) \cong R_{\Lambda\cap\Gamma}\Big/\left(\sum_{\ell\in\Lambda} I_\ell\Big/\left(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\right)\right),$$

one has the natural isomorphism

$$\kappa\colon \operatorname{Hom}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_{\Lambda\cap\Gamma}) \longrightarrow \left(0 :_{R_{\Lambda\cap\Gamma}} \sum_{\ell\in\Lambda} I_\ell\Big/\left(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\right)\right),$$

$\kappa(\psi) = \psi(\dot\alpha)$, where $\dot\alpha = (\dot\alpha_{\mathbf{i}})_{\mathbf{i}\subseteq[n]\setminus\Lambda}$ with

$$\dot\alpha_{\mathbf{i}} = \begin{cases} 0, & \text{if } \mathbf{i}\neq\emptyset,\\ 1, & \text{if } \mathbf{i}=\emptyset,\end{cases}$$

is the identity element of R_Λ.

Next we show that

$$\left(0 :_{R_{\Lambda\cap\Gamma}} \sum_{\ell\in\Lambda} I_\ell\Big/\left(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\right)\right) = \sum_{\ell\in\Lambda} I_\ell\Big/\left(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\right).$$

Set $\Lambda\setminus\Gamma = \{a\}$. Let $\gamma = (\gamma_{\mathbf{i}})_{\mathbf{i}\subseteq[n]\setminus\Lambda\cap\Gamma}$ be an element of

$$\left(0 :_{R_{\Lambda\cap\Gamma}} \sum_{\ell\in\Lambda} I_\ell\Big/\left(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\right)\right).$$

If $\gamma \notin \sum_{\ell\in\Lambda} I_\ell/\left(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\right)$, then there exists $\mathbf{v}\subseteq[n]\setminus\Lambda\cap\Gamma$ such that $a\notin\mathbf{v}$ and $\gamma_{\mathbf{v}}\neq 0$. Set $M = R\gamma_{\mathbf{v}}$, which is a non-zero submodule of $B_{\mathbf{v}}$. As B_a is a semidualizing R-module and $M\neq 0$, we have $B_a\otimes_R M\neq 0$. Thus there exists an element e of B_a such that $e\otimes\gamma_{\mathbf{v}}\neq 0$. Set $\theta = (\theta_{\mathbf{i}})_{\mathbf{i}\subseteq[n]\setminus\Lambda\cap\Gamma}$, where

$$\theta_{\mathbf{i}} = \begin{cases} 0, & \text{if } \mathbf{i}\neq\{a\},\\ e, & \text{if } \mathbf{i}=\{a\}.\end{cases}$$

Note that θ is an element of $\sum_{\ell\in\Lambda} I_\ell/\big(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\big)$ and $\gamma\theta \neq 0$, which contradicts with $\gamma \in \big(0 :_{R_{\Lambda\cap\Gamma}} \sum_{\ell\in\Lambda} I_\ell/\big(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\big)\big)$. Therefore

$$\left(0 :_{R_{\Lambda\cap\Gamma}} \sum_{\ell\in\Lambda} I_\ell\Big/\Big(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\Big)\right) \subseteq \sum_{\ell\in\Lambda} I_\ell\Big/\Big(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\Big).$$

On the other hand $\sum_{\ell\in\Lambda} I_\ell/\big(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\big) \subseteq \big(0 :_{R_{\Lambda\cap\Gamma}} \sum_{\ell\in\Lambda} I_\ell/\big(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\big)\big)$. Indeed, if $\alpha = (\alpha_{\mathbf{i}})_{\mathbf{i}\subseteq[n]\setminus\Lambda\cap\Gamma}$ and $\alpha' = (\alpha'_{\mathbf{i}})_{\mathbf{i}\subseteq[n]\setminus\Lambda\cap\Gamma}$ are two elements of $\sum_{\ell\in\Lambda} I_\ell/\big(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\big)$, then $\alpha_{\mathbf{i}} = 0 = \alpha'_{\mathbf{i}}$ for all $\mathbf{i}$ such that $a \notin \mathbf{i}$. Hence, by Lemma 3.11, $\alpha\alpha' = 0$. Thus

$$\kappa\colon \mathrm{Hom}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_{\Lambda\cap\Gamma}) \longrightarrow \sum_{\ell\in\Lambda} I_\ell\Big/\Big(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\Big), \quad \kappa(\psi) = \psi(\dot{\alpha}) \tag{6}$$

is an $R_{\Lambda\cap\Gamma}$-isomorphism.

By (4), $\text{G-dim}_{R_{\Lambda\cap\Gamma}} R_\Lambda = 0$. Let $\mathbf{F}$ be a minimal free resolution of R_Λ over $R_{\Lambda\cap\Gamma}$. Note that $\sum_{\ell\in\Lambda} I_\ell/\big(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\big)$ is the first syzygy of R_Λ in $\mathbf{F}$. By [1, Construction 3.6] and (6), we can construct a Tate resolution of R_Λ as $\mathbf{T} \to \mathbf{F} \to R_\Lambda$, where $\mathbf{T}$ construct by splicing $\mathbf{F}$ with $\mathrm{Hom}_{R_{\Lambda\cap\Gamma}}(\mathbf{F}, R_{\Lambda\cap\Gamma})$. Hence $\mathbf{T} \cong \mathrm{Hom}_{R_{\Lambda\cap\Gamma}}(\mathbf{T}, R_{\Lambda\cap\Gamma})$. This explains the first isomorphism in the next sequence

$$\begin{aligned}\widehat{\mathrm{Tor}}_i^{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) &= \mathrm{H}_i(\mathbf{T} \otimes_{R_{\Lambda\cap\Gamma}} R_\Gamma)\\ &\cong \mathrm{H}_i\big(\mathrm{Hom}_{R_{\Lambda\cap\Gamma}}(\mathbf{T}, R_{\Lambda\cap\Gamma}) \otimes_{R_{\Lambda\cap\Gamma}} R_\Gamma\big)\\ &\cong \mathrm{H}_i\big(\mathrm{Hom}_{R_{\Lambda\cap\Gamma}}(\mathbf{T}, R_\Gamma)\big)\\ &= \widehat{\mathrm{Ext}}^{-i}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma),\end{aligned} \tag{7}$$

for all $i \in \mathbb{Z}$. As each $R_{\Lambda\cap\Gamma}$-module $\mathbf{T}_i$ is finite and free, the second isomorphism follows.

By (4), $\text{G-dim}_{R_{\Lambda\cap\Gamma}} R_\Lambda = 0$ and so, by [1, Theorem 5.2], one has

$$\widehat{\mathrm{Tor}}_i^{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) \cong \mathrm{Tor}_i^{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) \quad \text{and} \quad \widehat{\mathrm{Ext}}^i_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) \cong \mathrm{Ext}^i_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma), \tag{8}$$

for all $i \geqslant 1$. Thus, by (7), (8) and (5), one gets

$$\begin{aligned}\widehat{\mathrm{Ext}}^{-i}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) &\cong \widehat{\mathrm{Tor}}_i^{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) \cong \mathrm{Tor}_i^{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) = 0,\\ \widehat{\mathrm{Tor}}_{-i}^{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) &\cong \widehat{\mathrm{Ext}}^i_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) \cong \mathrm{Ext}^i_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) = 0,\end{aligned}$$

for all $i \geqslant 1$. Therefore, by (7), to complete the proof it is enough to show that $\widehat{\operatorname{Ext}}^{0}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) = 0$. As $\widehat{\operatorname{Ext}}^{-1}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) = 0$ and R_Λ is totally reflexive as an $R_{\Lambda\cap\Gamma}$-module one has, by [1, Lemma 5.8], the exact sequence

$$0 \to \operatorname{Hom}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_{\Lambda\cap\Gamma}) \otimes_{R_{\Lambda\cap\Gamma}} R_\Gamma \xrightarrow{\nu} \operatorname{Hom}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) \longrightarrow \widehat{\operatorname{Ext}}^{0}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) \to 0, \quad (9)$$

where the map ν is given by

$$\nu(\psi \otimes \theta) = \psi_\theta, \quad \psi_\theta(\alpha) = \psi(\alpha)\theta.$$

In a similar way to (6), one gets the natural isomorphism $\tau\colon \operatorname{Hom}_{R_\Gamma}(R_{\Lambda\cup\Gamma}, R_\Gamma) \longrightarrow \sum_{\ell\in\Lambda\cup\Gamma} I_\ell / \big(\sum_{\ell\in\Gamma} I_\ell\big)$ given by $\tau(\psi) = \psi(\dot{\varphi})$, where $\dot{\varphi}$ is the identity element of $R_{\Lambda\cup\Gamma}$. It is straightforward to show that the following diagram commutes:

$$\begin{array}{ccc}
\operatorname{Hom}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_{\Lambda\cap\Gamma}) \otimes_{R_{\Lambda\cap\Gamma}} R_\Gamma & \xrightarrow[\nu]{} & \operatorname{Hom}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) \\
{\scriptstyle \kappa\otimes R_\Gamma}\big\downarrow \cong & & {\scriptstyle f}\big\downarrow \cong \\
\sum_{\ell\in\Lambda} I_\ell / \big(\sum_{\ell\in\Lambda\cap\Gamma} I_\ell\big) \otimes_{R_{\Lambda\cap\Gamma}} R_\Gamma & & \operatorname{Hom}_{R_\Gamma}(R_{\Lambda\cup\Gamma}, R_\Gamma) \\
{\scriptstyle g}\big\downarrow \cong & & {\scriptstyle \tau}\big\downarrow \cong \\
I_a / \big(\sum_{\ell\in\Gamma} I_a I_\ell\big) & \xrightarrow[h]{\cong} & \sum_{\ell\in\Lambda\cup\Gamma} I_\ell / \big(\sum_{\ell\in\Gamma} I_\ell\big)
\end{array}$$

where the maps f, g and h are natural isomorphisms. Hence ν is surjective and (9) implies that $\widehat{\operatorname{Ext}}^{0}_{R_{\Lambda\cap\Gamma}}(R_\Lambda, R_\Gamma) = 0$.

The following results give a partial converse to Theorem 3.9. Note that Proposition 3.16 is a generalization of the result of Jorgensen et al. [11, Theorem 3.1].

PROPOSITION 3.15. *Let R be a Cohen-Macaulay ring. Assume that there exist a Gorenstein local ring Q and ideals $I_1, \ldots, I_n$ of Q satisfying the following conditions:*

(1) *there is a ring isomorphism $R \cong Q/(I_1 + \cdots + I_n)$,*

(2) *the ring $R_k = Q/(I_1 + \cdots + I_k)$ is Cohen-Macaulay for all $k \in [n]$,*

(3) $\operatorname{fd}_{R_j}(R_k) < \infty$ *for all $k \in [n]$ and all $1 \leqslant j \leqslant k$,*

(4) *for each $k \in [n]$ and all $0 \leqslant j < k$, $\mathrm{I}^{R_k}_{R_k}(t) \neq t^e \mathrm{I}^{R_j}_{R_j}(t)$ for any integer e, ($R_0 = Q$).*

Then there exist integers $g_0, g_1, \dots, g_{n-1}$ such that

$$[\mathrm{Ext}_Q^{g_0}(R, Q)] \lhd [\mathrm{Ext}_{R_1}^{g_1}(R, R_1)] \lhd \cdots \lhd [\mathrm{Ext}_{R_{n-1}}^{g_{n-1}}(R, R_{n-1})] \lhd [R]$$

is a chain in $\mathfrak{G}_0(R)$ of length n.

Proof. We prove by induction. For $n = 1$, it is clear that $\mathrm{Ext}_Q^{g_0}(R, Q)$ is a dualizing R-module for some integer g_0. It will be shown in following that condition (4) implies $[\mathrm{Ext}_Q^{g_0}(R, Q)] \lhd [R]$. Let $n = 2$. As $\mathrm{fd}_{R_1}(R) < \infty$, one has $\text{G-dim}_{R_1}(R) < \infty$. Then, by Remark 2.2, there exists an integer g_1 such that $\mathrm{Ext}_{R_1}^i(R, R_1) = 0$ for all $i \neq g_1$ and $C_1 = \mathrm{Ext}_{R_1}^{g_1}(R, R_1)$ is a semidualizing R-module. Therefore there is an isomorphism $C_1 \simeq \Sigma^{g_1} \mathbf{R}\mathrm{Hom}_{R_1}(R, R_1)$ in the derived category $\mathrm{D}(R)$. Thus, by [2, (1.7.8)], $\mathrm{I}_R^{C_1}(t) = t^{-g_1}\mathrm{I}_{R_1}^{R_1}(t)$. Also there exists an integer g_0 such that $\mathrm{Ext}_Q^i(R, Q) = 0$ for all $i \neq g_0$ and $D = \mathrm{Ext}_Q^{g_0}(R, Q)$ is a dualizing R-module and then $D \simeq \Sigma^{g_0} \mathbf{R}\mathrm{Hom}_Q(R, Q)$ in $\mathrm{D}(R)$. Assumption (4) implies that C_1 is a non-trivial semidualizing R-module and so $[D] \lhd [C_1] \lhd [R]$ is a chain in $\mathfrak{G}_0(R)$ of length 2.

Let $n > 2$ and suppose that the assertion holds true for $n - 1$. By induction there exist integers $h_0, h_1, \dots, h_{n-2}$ such that

$$[\mathrm{Ext}_Q^{h_0}(R_{n-1}, Q)] \lhd [\mathrm{Ext}_{R_1}^{h_1}(R_{n-1}, R_1)] \lhd \cdots \lhd [\mathrm{Ext}_{R_{n-2}}^{h_{n-2}}(R_{n-1}, R_{n-2})] \lhd [R_{n-1}] \tag{10}$$

is a chain in $\mathfrak{G}_0(R_{n-1})$ of length $n - 1$. (In fact, there is an isomorphism $\mathrm{Ext}_{R_i}^{h_i}(R_{n-1}, R_i) \simeq \Sigma^{h_i} \mathbf{R}\mathrm{Hom}_{R_i}(R_{n-1}, R_i)$ in $\mathrm{D}(R_{n-1})$, for all $0 \leqslant i \leqslant n-2$.)

As $\mathrm{fd}_{R_k}(R) < \infty$, one has $\text{G-dim}_{R_k}(R) < \infty$, for all $k \in [n]$, and so, by Remark 2.2, there exists an integer g_k such that $\mathrm{Ext}_{R_k}^i(R, R_k) = 0$, for all $i \neq g_k$, and $C_k = \mathrm{Ext}_{R_k}^{g_k}(R, R_k)$ is a semidualizing R-module. We have $C_k \simeq \Sigma^{g_k} \mathbf{R}\mathrm{Hom}_{R_k}(R, R_k)$ in $\mathrm{D}(R)$. Also there exists an integer g_0 such that $\mathrm{Ext}_Q^i(R, Q) = 0$, for all $i \neq g_0$, and $D = \mathrm{Ext}_Q^{g_0}(R, Q)$ is a dualizing for R and so $D \simeq \Sigma^{g_0} \mathbf{R}\mathrm{Hom}_Q(R, Q)$ in $\mathrm{D}(R)$. Note that there is an isomorphism $\mathbf{R}\mathrm{Hom}_{R_k}(R, R_k) \simeq \mathbf{R}\mathrm{Hom}_{R_{n-1}}(R, \mathbf{R}\mathrm{Hom}_{R_k}(R_{n-1}, R_k))$, $0 \leqslant k \leqslant n-1$, in $\mathrm{D}(R)$, and R is a finite R_{n-1}-module with $\mathrm{fd}_{R_{n-1}}(R) < \infty$. Thus, by [5, Theorem 5.7] and (10), one obtains $[\mathrm{Ext}_{R_{k-1}}^{g_{k-1}}(R, R_{k-1})] \trianglelefteq [\mathrm{Ext}_{R_k}^{g_k}(R, R_k)]$, for all $1 \leqslant k \leqslant n-1$. By [2, (1.7.8)], $\mathrm{I}_R^{C_k}(t) = t^{-g_k}\mathrm{I}_{R_k}^{R_k}(t)$ for all $1 \leqslant k \leqslant n-1$ and $\mathrm{I}_R^D(t) = t^{-g_0}\mathrm{I}_Q^Q(t)$. Therefore, by condition (4), $[\mathrm{Ext}_{R_{k-1}}^{g_{k-1}}(R, R_{k-1})] \lhd [\mathrm{Ext}_{R_k}^{g_k}(R, R_k)]$ for all $1 \leqslant k \leqslant n-1$, and $[\mathrm{Ext}_{R_{n-1}}^{g_{n-1}}(R, R_{n-1})] \lhd [R]$. Hence

$$[\mathrm{Ext}_Q^{g_0}(R, Q)] \lhd [\mathrm{Ext}_{R_1}^{g_1}(R, R_1)] \lhd \cdots \lhd [\mathrm{Ext}_{R_{n-1}}^{g_{n-1}}(R, R_{n-1})] \lhd [R]$$

is a chain in $\mathfrak{G}_0(R)$ of length n.

PROPOSITION 3.16. *Let R be a Cohen-Macaulay ring. Assume that there exist a Gorenstein local ring Q and ideals $I_1, \dots, I_n$ of Q satisfying the following conditions:*

(1) *there is a ring isomorphism $R \cong Q/(I_1 + \cdots + I_n)$,*

(2) *for each $\Lambda \subseteq [n]$, the ring $R_\Lambda = Q/\big(\sum_{\ell\in\Lambda} I_\ell\big)$ is Cohen-Macaulay,*

(3) *for subsets Λ, Γ of $[n]$ with $\Lambda \cap \Gamma = \emptyset$,*
   (i) $\operatorname{Tor}^Q_{\geqslant 1}(R_\Lambda, R_\Gamma) = 0$,
   (ii) *for all $i \in \mathbb{Z}$, $\widehat{\operatorname{Ext}}^i_Q(R_\Lambda, R_\Gamma) = 0 = \widehat{\operatorname{Tor}}^Q_i(R_\Lambda, R_\Gamma)$,*

(4) *for two subsets Λ, Γ of $[n]$ with $\Lambda \neq \Gamma$ and for any integer e, $\mathrm{I}^{R_\Lambda}_{R_\Lambda}(t) \neq t^e \mathrm{I}^{R_\Gamma}_{R_\Gamma}(t)$.*

Then, for each $\Lambda \subseteq [n]$, there is an integer g_Λ such that $\operatorname{Ext}^{g_\Lambda}_{R_\Lambda}(R, R_\Lambda)$ is a semidualizing R-module. As conclusion, R admits 2^n non-isomorphic semidualizing modules.

PROOF. For two subsets Λ, Γ of $[n]$ with $\Gamma \subseteq \Lambda$, we have $\text{G-dim}_{R_\Gamma}(R_\Lambda) < \infty$. Indeed, $\text{G-dim}_Q(R_{\Lambda\setminus\Gamma}) < \infty$, since Q is Gorenstein. Thus $R_{\Lambda\setminus\Gamma}$ admits a Tate resolution $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} R_{\Lambda\setminus\Gamma}$ over Q, where ϑ_i is isomorphism for all $i \gg 0$. We show that the induced diagram $\mathbf{T}\otimes_Q R_\Gamma \xrightarrow{\vartheta\otimes_Q R_\Gamma} \mathbf{P}\otimes_Q R_\Gamma \xrightarrow{\pi\otimes_Q R_\Gamma} R_{\Lambda\setminus\Gamma} \otimes_Q R_\Gamma$ is a Tate resolution of $R_{\Lambda\setminus\Gamma} \otimes_Q R_\Gamma \cong R_\Lambda$ over R_Γ. By condition (3)(i), $\mathbf{P} \otimes_Q R_\Gamma$ is a free resolution of R_Λ over R_Γ. Also by assumption, $\widehat{\operatorname{Tor}}^Q_i(R_{\Lambda\setminus\Gamma}, R_\Gamma) = 0$, for all $i \in \mathbb{Z}$, and then $\mathbf{T} \otimes_Q R_\Gamma$ is an exact complex of finite free R_Γ-modules. Of course, the map $\vartheta_i \otimes_Q R_\Gamma$ is an isomorphism, for all $i \gg 0$. In order to show that $\operatorname{Hom}_{R_\Gamma}(\mathbf{T} \otimes_Q R_\Gamma, R_\Gamma)$ is exact we note that the sequence of isomorphisms

$$\operatorname{Hom}_{R_\Gamma}(\mathbf{T} \otimes_Q R_\Gamma, R_\Gamma) \cong \operatorname{Hom}_Q(\mathbf{T}, \operatorname{Hom}_{R_\Gamma}(R_\Gamma, R_\Gamma)) \cong \operatorname{Hom}_Q(\mathbf{T}, R_\Gamma),$$

implies that

$$\mathrm{H}_i(\operatorname{Hom}_{R_\Gamma}(\mathbf{T} \otimes_Q R_\Gamma, R_\Gamma)) \cong \mathrm{H}_i(\operatorname{Hom}_Q(\mathbf{T}, R_\Gamma)) \cong \widehat{\operatorname{Ext}}^{-i}_Q(R_{\Lambda\setminus\Gamma}, R_\Gamma),$$

which is zero, by condition (3)(ii), for all $i \in \mathbb{Z}$. Hence the complex $\operatorname{Hom}_{R_\Gamma}(\mathbf{T} \otimes_Q R_\Gamma, R_\Gamma)$ is exact and so R_Λ admits a Tate resolution over R_Γ. Therefore $\text{G-dim}_{R_\Gamma}(R_\Lambda) < \infty$.

In particular, $\text{G-dim}_{R_\Lambda}(R) < \infty$, for all $\Lambda \subseteq [n]$. Hence, by Remark 2.2, $\operatorname{Ext}^i_{R_\Lambda}(R, R_\Lambda) = 0$ for all $i \neq g_\Lambda$, where $g_\Lambda := \text{G-dim}_{R_\Lambda}(R)$, and $C_\Lambda := \operatorname{Ext}^{g_\Lambda}_{R_\Lambda}(R, R_\Lambda)$ is a semidualizing R-module.

Note that there is an isomorphism $C_\Lambda \simeq \Sigma^{g_\Lambda} \mathbf{R}\mathrm{Hom}_{R_\Lambda}(R, R_\Lambda)$ in the derived category $\mathrm{D}(R)$. Therefore, by [2, (1.7.8)],

$$\mathrm{I}_R^{C_\Lambda}(t) = \mathrm{I}_R^{\Sigma^{g_\Lambda} \mathbf{R}\mathrm{Hom}_{R_\Lambda}(R, R_\Lambda)}(t) = t^{-g_\Lambda} \mathrm{I}_{R_\Lambda}^{R_\Lambda}(t).$$

Now condition (4) implies that the 2^n semidualizing C_Λ are pairwise non-isomorphic.

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