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POSITIVE SOLUTIONS FOR PARAMETRIC SEMILINEAR ROBIN PROBLEMS WITH INDEFINITE AND UNBOUNDED POTENTIAL

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Abstract

We consider a parametric Robin problem driven by the Laplace operator plus an indefinite and unbounded potential. The reaction term is a Carathéodory function which exhibits superlinear growth near $+\infty$ without satisfying the Ambrosetti-Rabinowitz condition. We are looking for positive solutions and prove a bifurcation-type theorem describing the dependence of the set of positive solutions on the parameter. We also establish the existence of the minimal positive solution u_λ^* and investigate the monotonicity and continuity properties of the map $\lambda \mapsto u_\lambda^*$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we deal with the following semilinear Robin problem:

$$\left\{\begin{aligned} &-\Delta u(z) + (\xi(z) + \lambda)u(z) = f(z, u(z)), && \text{in } \Omega, \\ &\frac{\partial u}{\partial n} + \beta(z)u = 0, && \text{on } \partial\Omega, \quad \lambda > 0, u > 0. \end{aligned}\right\} \qquad (P_\lambda)$$

In this problem, the potential function $\xi \in L^s(\Omega)$ with $s > N$, when $N \geq 2$, and $s = 1$, when $N = 1$, and is indefinite (that is, changes sign) and unbounded below, while $\lambda > 0$ is a parameter. The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$ the map $x \mapsto f(z, x)$ is continuous), which exhibits superlinear growth near $+\infty$ but without satisfying the usual, in such cases, Ambrosetti-Rabinowitz condition (the AR-condition for short). In the boundary condition, $\partial u/\partial n$ denotes the normal derivative of u defined by

$$\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}, \quad \text{for all } u \in H^2(\Omega),$$

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with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary weight function $\beta \in W^{1,\infty}(\partial\Omega)$ satisfies $\beta \geq 0$. Evidently, the case $\beta \equiv 0$ corresponds to the usual Neumann problem.

We are looking for positive solutions and our aim is to determine the precise dependence of the set of positive solutions of problem (P_λ) on the parameter $\lambda > 0$. So, we prove a bifurcation-type result, establishing the existence of a critical parameter value $\lambda_* > 0$ such that

- for all $\lambda > \lambda_*$, problem (P_λ) has at least two positive solutions;
- for $\lambda = \lambda_*$, problem (P_{λ_*}) has at least one positive solution;
- for $\lambda \in (0, \lambda_*)$, problem (P_λ) has no positive solutions.

Recently existence and multiplicity theorem for semilinear problems with indefinite and unbounded potential, were proved by Kyritsi and Papageorgiou [6], Li and Wang [7], Papageorgiou and Papalini [10], Qin, Tang and Zhang [16], Zhang and Liu [18] (Dirichlet problems), Papageorgiou and Rădulescu [11], [13], Papageorgiou and Smyrlis [15] (Neumann problems) and Papageorgiou and Rădulescu [12], [14] (Robin problems). However, none of the aforementioned works focuses on positive solutions or on their dependence on the parameter $\lambda > 0$ of the problem. These works with the exception of [15], deal with nonparametric equations under resonance conditions and prove existence and multiplicity theorems. Papageorgiou and Smyrlis [15] examine Neumann problems (that is, $\beta \equiv 0$) with an indefinite and unbounded potential (exactly as in this work) with a reaction term of logistic type. More precisely, their reaction term has the form

$$\lambda x - f(z, x),$$

with $\lambda > 0$ being the parameter and $f(z, x)$ being a Carathéodory function that exhibits superlinear growth near $\pm\infty$. They show that, for all $\lambda > \hat{\lambda}_2$ ($\hat{\lambda}_2$ being the second eigenvalue of the differential operator $u \mapsto -\Delta u + \xi(z)u$) these problems admit multiple solutions for which they provide sign information. In the present paper, the setting is complementary since the parameter appears with a negative sign in the reaction term while the perturbation $f(z, \cdot)$ is superlinear near $\pm\infty$ (without satisfying the well-known Ambrosetti-Rabinowitz condition). Here we focus on positive solutions and establish the precise dependence of these solutions on the parameter $\lambda > 0$. We point out that our formulation in this paper includes as a special case Neumann problems (they correspond to $\beta \equiv 0$). Also, it is worth mentioning that although we have a problem with different geometry than considered in [15], nevertheless we have existence and multiplicity of the positive solutions for large values of the parameter $\lambda > 0$. In this sense, we can say that our problem exhibits bifurcation-type properties near infinity. So, we have a situation complementary to the well-known and extensively studied case of convex-concave problems (problems with competing nonlinearities) for which bifurcation occurs near zero.

Our tools are variational and are based on critical point theory together with suitable truncation, perturbation and comparison techniques. In the next section for the convenience of the reader, we briefly review some of those tools.

2. Mathematical background

Let X be a Banach space and X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following holds:

"every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the functional $\varphi \in C^1(X, \mathbb{R})$, which leads to a deformation theorem for the sublevel sets of φ, from which one can derive the minimax theory of the critical values of φ. Prominent in that theory is the so-called "mountain pass theorem" due to Ambrosetti and Rabinowitz [2], which we state here in a slightly more general form (see, for example, Gasinski and Papageorgiou [4, p. 648]).

THEOREM 2.1. *Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$ with $\|u_1 - u_0\| > \rho$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : \|u - u_0\| = \rho] = m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}.$$

Then $c \geq m_\rho$ and c is a critical value of φ.

In the analysis of problem (P_λ), we will use the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^q(\partial\Omega)$ with $1 \leq q \leq \infty$. In the sequel, by $\|\cdot\|$ we denote the norm of the Sobolev space $H^1(\Omega)$ defined by

$$\|u\| = \left[\|u\|_2^2 + \|Du\|_2^2\right]^{1/2}, \quad \text{for all } u \in H^1(\Omega).$$

The space $C^1(\overline{\Omega})$ becomes an ordered Banach space with the order induced by the following order cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0, \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior in the $C(\overline{\Omega})$-topology given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0, \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure which we denote by $\sigma(\,\cdot\,)$. This measure permits the introduction of the boundary Lebesgue spaces $L^q(\partial\Omega)$, $1 \leq q \leq \infty$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0\colon H^1(\Omega) \to L^2(\partial\Omega)$, known as the "trace map" such that

$$\gamma_0(u) = u|_{\partial\Omega}, \quad \text{for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

So, we understand the trace map as representing the "boundary values" of a Sobolev function. The trace map $\gamma_0(\,\cdot\,)$ is compact into $L^q(\partial\Omega)$ for every $q \in \left[1, \frac{2(N-1)}{N-2}\right)$ and, in addition, we have

$$\operatorname{im}\gamma_0 = H^{1/2,2}(\partial\Omega) \quad \text{and} \quad \ker\gamma_0 = H_0^1(\Omega).$$

In what follows, for the sake of notational simplicity, we drop the use of the trace map. The restrictions of all Sobolev functions on $\partial\Omega$, are understood in the sense of traces.

Suppose that $f_0\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f_0(z,x)| \leq a_0(z)(1 + |x|^{r-1}), \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$ and

$$1 < r < 2^* = \begin{cases} 2N/(N-2), & \text{if } N \geq 3, \\ +\infty, & \text{if } N = 1, 2. \end{cases}$$

We set $F_0(z,x) = \int_0^x f_0(z,s)\,ds$, $\vartheta(u) = \|Du\|_2^2 + \int_\Omega \xi(z)u^2\,dz + \int_{\partial\Omega} \beta(z)u^2\,d\sigma$, for all $H^1(\Omega)$, and consider the C^1-functional $\varphi_0\colon H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{2}\vartheta(u) - \int_\Omega F_0(z,u)\,dz, \quad \text{for all } u \in H^1(\Omega).$$

From Papageorgiou and Rădulescu [12], we have the following result relating local minimizers of φ_0. We assume that:

- $\xi \in L^s(\Omega)$ with $s > N$, if $N \geq 2$, or $s = 1$, if $N = 1$;
- $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta \geq 0$;
- $f_0(z, x)$ is a Carathéodory function as above.

PROPOSITION 2.2. *Assume that $u_0 \in H^1(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h), \quad \textit{for all } h \in C^1(\overline{\Omega}) \textit{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0.$$

Then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and u_0 is also a local $H^1(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h), \quad \textit{for all } h \in H^1(\Omega) \textit{ with } \|h\| \leq \rho_1.$$

From Papageorgiou and Rădulescu [14], we know that there exist $\mu > 0$ and $c_0 > 0$ such that

$$\vartheta(u) + \mu \|u\|_2^2 \geq c_0 \|u\|^2, \quad \text{for all } u \in H^1(\Omega). \tag{1}$$

Recall that a Banach space X has the so-called "Kadec-Klee property" if the following holds:

$$\text{“}u_n \xrightarrow{w} u \text{ in } X \text{ and } \|u_n\| \to \|u\| \implies u_n \to u \text{ in } X\text{”}.$$

Using the parallelogram law, we see that every Hilbert space has the Kadec-Klee property.

Given $x \in \mathbb{R}$, let $x^\pm = \max\{\pm x, 0\}$. For $u \in H^1(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u^\pm \in H^1(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Given $\varphi \in C^1(X, \mathbb{R})$, by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Also, if $\eta \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function), we define

$$N_\eta(u)(\cdot) = \eta(\cdot, u(\cdot)), \quad \text{for all } u \in H^1(\Omega),$$

the Nemytskii map corresponding to the function $\eta(z, x)$.

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Consider the following Robin eigenvalue problem:

$$\begin{Bmatrix} -\Delta u(z) + \xi(z)u(z) = \hat{\lambda} u(z), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0, & \text{on } \partial\Omega. \end{Bmatrix}$$

We know that this problem has a smallest eigenvalue $\hat{\lambda}_1$ (which may be negative) and $\hat{\lambda}_1$ is simple with eigenfunctions that do not change sign. Moreover, if $\xi \in L^{N/2}(\Omega)$ and $\beta \in L^\infty(\partial\Omega)$, we have

$$\hat{\lambda}_1 = \inf\left[\frac{\vartheta(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0\right].$$

The infimum is realized on the corresponding one-dimensional eigenspace. If $\xi \in L^s(\Omega)$ with $s > N$ and $\beta \in W^{1,\infty}(\partial\Omega)$, then the eigenfunctions belong to $C^1(\overline{\Omega})$ (see Wang [17]). Let $\hat{u}_1$ be the positive, L^2-normalized (that is, $\|\hat{u}_1\|_2 = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. If in addition $\xi^+ \in L^\infty(\Omega)$, then $\hat{u}_1 \in \operatorname{int} C_+$ (see Papageorgiou and Rădulescu [12], [14]).

Also, by $A \in \mathscr{L}(H^1(\Omega), H^1(\Omega)^*)$ we denote the linear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N}\, dz, \quad \text{for all } u, h \in H^1(\Omega).$$

Finally if $p \in [1 + \infty)$, then $p' \in (1, +\infty]$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

3. Bifurcation-type theorem

In this section, we prove a bifurcation-type theorem describing the dependence of the set of positive solutions of problem (P_λ) on the parameter $\lambda > 0$.

Our hypotheses on the data of problem (P_λ) are the following:

$H(\xi)$: $\xi \in L^s(\Omega)$ with $s > N$, if $N \geq 2$, $s = 1$, if $N = 1$, and $\xi^+ \in L^\infty(\Omega)$.

$H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geq 0$, for all $z \in \partial\Omega$.

REMARK 3.1. By taking $\beta \equiv 0$, we see that we cover also the Neumann problem.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$, for almost all $z \in \Omega$, and

(i) $|f(z, x)| \leq a(z)(1 + x^{r-1})$, for almost all $z \in \Omega$, all $x \geq 0$, with $a \in L^\infty(\Omega)$, $2 < r < 2^*$;

(ii) if $F(z,x) = \int_0^x f(z,x)\,ds$, then $\lim_{x\to+\infty} F(z,x)/x^2 = +\infty$, uniformly for almost all $z \in \Omega$;

(iii) there exists $\tau \in (\max\{1, (r-2)N/2\}, 2^*)$ such that

$$0 < \gamma_0 \le \liminf_{x\to+\infty} \frac{f(z,x)x - 2F(z,x)}{|x|^\tau},$$

uniformly for almost all $z \in \Omega$;

(iv) there exist $\delta_0 > 0$, $d \in (1,2)$ and $\eta \in L^\infty(\Omega)$ such that

* $\hat{c}x^{d-1} \le f(z,x)$, for almost all $z \in \Omega$, all $x \in [0,\delta_0]$, with $\hat{c} > 0$,
* $\hat{\lambda}_1 \le \eta(z)$, for almost all $z \in \Omega$, the inequality is strict on a set of positive measure, and
* $\eta(z)x \le f(z,x)$, for almost all $z \in \Omega$, all $x \ge 0$.

REMARK 3.2. Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality, we may assume that $f(z,x) = 0$, for almost all $z \in \Omega$, all $x \le 0$. Hypotheses $H(f)$(ii) and (iii) imply that

$$\lim_{x\to+\infty} \frac{f(z,x)}{x} = +\infty, \quad \text{uniformly for almost all } z \in \Omega,$$

that is, in problem (P_λ) the reaction term $f(z,\cdot)$ is superlinear near $+\infty$. These two hypotheses, are weaker than the usual AR-condition (unilateral version) which says that there exist $M > 0$ and $q > 2$ such that

$$0 < qF(z,x) \le f(z,x)x, \text{ for almost all } z \in \Omega, \text{ all } x \ge M, \tag{2a}$$

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M) \tag{2b}$$

(see Ambrosetti and Rabinowitz [2] and Mugnai [9]). Integrating (2a) and using (2b), we obtain the weaker condition

$$c_1 x^q \le F(z,x), \quad \text{for almost all } z \in \Omega, \text{ all } x \ge M. \tag{3}$$

So, the AR-condition implies that $f(z,\cdot)$ has at least $(q-1)$-polynomial growth (see relations (3) and (2a)). No such restriction is imposed on $f(z,\cdot)$ by hypothesis $H(f)$. In this way we incorporate in our framework superlinear functions with "slower" growth near $+\infty$, which fail to satisfy the AR-condition (2a), (2b). To see this, consider the following function (for the sake of simplicity we drop the z-dependence):

$$f(x) = \begin{cases} cx^{d-1}, & \text{if } 0 \le x \le 1, \\ 2cx\left(\ln x + \frac{1}{2}\right), & \text{if } 1 < x, \end{cases}$$

with $d \in (1, 2)$, $c > \max\{0, \hat{\lambda}_1\}$. This function satisfies hypothesis $H(f)$ but fails to satisfy the AR-condition. On the other hand, the function $f(x) = x^{r-1} + \hat{c}x^{d-1}$, for all $x \geq 0$, with $1 < d < 2 < r < 2^*$ and $\hat{c} > \max\{0, \hat{\lambda}_1\}$ satisfies hypothesis $H(f)$ and the AR-condition.

We introduce the following two sets:

$$\mathscr{L} = \{\lambda > 0 : \text{problem } (P_\lambda) \text{ has a positive solution}\}$$

(this is the set of admissible parameters),

$$S(\lambda) = \text{the set of positive solutions for problem } (P_\lambda)$$

(if $\lambda \notin \mathscr{L}$, then $S(\lambda) = \emptyset$).

Proposition 3.3. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then, for every $\lambda > 0$, $S(\lambda) \subseteq \operatorname{int} C_+$.*

Proof. We assume that $\lambda \in \mathscr{L}$ (otherwise $S(\lambda) = \emptyset$). Then we have $u \in S(\lambda)$ and

$$\left\{\begin{array}{ll} -\Delta u(z) + (\xi(z) + \lambda)u(z) = f(z, u(z)), & \text{for almost all } z \in \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0, & \text{on } \partial\Omega, \end{array}\right\} \tag{4}$$

see Papageorgiou and Rădulescu [12].

We define

$$k(z) = \begin{cases} 0, & \text{if } u(z) < 1, \\ \dfrac{f(z, u(z))}{u(z)} - \xi(z), & \text{if } 1 \leq u(z), \end{cases}$$

and

$$\ell(z) = \begin{cases} f(z, u(z)) - \xi(z)u(z), & \text{if } u(z) \leq 1, \\ 0, & \text{if } 1 < u(z). \end{cases}$$

Note that $k \in L^s(\Omega)$ (see hypotheses $H(\xi)$ and $H(f)$(i)) and $\ell \in L^\infty(\Omega)$. From (4) we have

$$\left\{\begin{array}{ll} -\Delta u(z) = (k(z) - \lambda)u(z) + \ell(z), & \text{for almost all } z \in \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0, & \text{on } \partial\Omega. \end{array}\right\} \tag{5}$$

From Lemma 5.1 of Wang [17], we have that $u \in L^\infty(\Omega)$. Then from (5) we see that $\Delta u \in L^s(\Omega)$. The Calderon-Zygmund estimates (see Wang [17, Lemma 5.2]) imply that $u \in W^{2,s}(\Omega)$ ($s > N$, when $N \geq 2$, see hypothesis

$H(\xi)$). By the Sobolev embedding theorem, we have $W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$, with $\alpha = 1 - N/s > 0$. Therefore $u \in C_+ \backslash \{0\}$.

Let $\rho = \|u\|_\infty$. Hypotheses $H(f)$(i) and (iv) imply that there exists $\hat{\xi}_\rho > 0$ such that

$$f(z, x) + \hat{\xi}_\rho x \geq 0, \quad \text{for almost all } z \in \Omega, \text{ all } 0 \leq x \leq \rho.$$

Then from (4) and the above inequality, we have

$$\begin{aligned} \Delta u(z) &\leq (\xi(z) + \lambda + \hat{\xi}_\rho) u(z), && \text{for almost all } z \in \Omega, \\ &\leq (\xi^+(z) + \lambda + \hat{\xi}_\rho) u(z), && \text{for almost all } z \in \Omega, \\ &\leq (\|\xi^+\|_\infty + \lambda + \hat{\xi}_\rho) u(z), && \text{for almost all } z \in \Omega \\ & && \text{(see hypothesis } H(\xi)), \\ \Longrightarrow \quad u \in \operatorname{int} C_+ \end{aligned}$$

(by the strong maximum principle, see Gasinski and Papageorgiou [4, p. 738]).

Therefore we have proved that

$$S(\lambda) \subseteq \operatorname{int} C_+, \quad \text{for all } \lambda > 0.$$

PROPOSITION 3.4. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold. Then $\mathscr{L} \neq \emptyset$ and $\lambda \in \mathscr{L}$ implies that $[\lambda, +\infty) \subseteq \mathscr{L}$.*

PROOF. Let $\mu > 0$ be as postulated by (1). We consider the following auxiliary Robin problem

$$\left\{ \begin{aligned} &-\Delta u(z) + (\xi(z) + \mu) u(z) = 1, && \text{in } \Omega, \\ &\frac{\partial u}{\partial n} + \beta(z) u(z) = 0, && \text{on } \partial\Omega, \ u > 0. \end{aligned} \right\} \tag{6}$$

Let $V \in \mathscr{L}(H^1(\Omega), H^1(\Omega)^*)$ be defined by

$$\langle V(u), h \rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} \, dz + \int_\Omega (\xi(z) + \mu) u h \, dz, \quad \text{for all } u, h \in H^1(\Omega).$$

Also, let $B \in \mathscr{L}(H^1(\Omega), L^2(\partial\Omega))$ be defined by

$$\langle B(u), h \rangle_{L^2(\partial\Omega)} = \int_{\partial\Omega} \beta(z) u h \, d\sigma, \quad \text{for all } u \in H^1(\Omega), \ h \in L^2(\partial\Omega).$$

Recall that γ_0 denotes the trace map and $\gamma_0 \in \mathscr{L}(H^1(\Omega), L^2(\partial\Omega))$. Then $\gamma_0^* \in \mathscr{L}(L^2(\partial\Omega), H^1(\Omega)^*)$. We consider the operator $K \in \mathscr{L}(H^1(\Omega), H^1(\Omega)^*)$ defined by

$$K(u) = V(u) + (\gamma_0^* \circ B)(u), \quad \text{for all } u \in H^1(\Omega).$$

We have

$$\langle K(u), u\rangle = \vartheta(u) + \mu \|u\|_2^2 \geq c_0 \|u\|^2, \text{ for all } u \in H^1(\Omega) \text{ (see (1))},$$
$$\Longrightarrow K(\cdot) \text{ is surjective}$$

(see, for example, Gasinski and Papageorgiou [4, p. 319]).

So, we can find $\bar{u} \in H^1(\Omega)$, $\bar{u} \neq 0$ such that

$$V(\bar{u}) + (\gamma_0^* \circ B)(\bar{u}) = 1. \tag{7}$$

On (7) we act with $-\bar{u}^- \in H^1(\Omega)$ and obtain

$$\begin{aligned} & \vartheta(\bar{u}^-) + \mu \left\|\bar{u}^-\right\|_2^2 \leq 0, \\ \Longrightarrow\ & c_0 \left\|\bar{u}^-\right\|_2^2 \leq 0, \quad \text{(see (1))} \\ \Longrightarrow\ & \bar{u} \geq 0, \quad \bar{u} \neq 0. \end{aligned}$$

From (7) we have

$$\int_\Omega (D\bar{u}, Dh)_{\mathbb{R}^N}\, dz + \int_\Omega (\xi(z) + \mu)\bar{u}h\, dz + \int_{\partial\Omega} \beta(z)\bar{u}h\, d\sigma$$
$$= \int_\Omega h\, dz, \quad \text{for all } h \in H^1(\Omega),$$

$$\Longrightarrow \left\{ \begin{array}{ll} -\Delta\bar{u}(z) + (\xi(z) + \mu)\bar{u}(z) = 1, & \text{for almost all } z \in \Omega, \\ \dfrac{\partial \bar{u}}{\partial n} + \beta(z)\bar{u} = 0, & \text{on } \partial\Omega \end{array} \right\} \tag{8}$$

(that is, $\bar{u}$ is a positive solution of (6)).

As before (see the proof of Proposition 3.3), using the regularity result of Wang [17], we show that $\bar{u} \in C_+\backslash\{0\}$. Also from (8) and hypothesis $H(\xi)$, we have

$$\Delta\bar{u}(z) \leq (\|\xi^+\|_\infty + \mu)u(z), \text{ for almost all } z \in \Omega, \implies \bar{u} \in \operatorname{int} C_+ \tag{9}$$

(by the strong maximum principle).

Let $\bar{m} = \min_{\overline{\Omega}} \bar{u} > 0$ (see (9)) and let $\lambda_0 = \mu + \|N_f(\bar{u})\|_\infty/\bar{m}$ (see hypothesis $H(f)$(i)). For every $h \in H^1(\Omega)$ with $h \geq 0$, we have

$$\int_\Omega (D\bar{u}, Dh)_{\mathbb{R}^N}\, dz + \int_\Omega (\xi(z) + \lambda_0)\bar{u}h\, dz + \int_{\partial\Omega} \beta(z)\bar{u}h\, d\sigma$$
$$= \int_\Omega (D\bar{u}, Dh)_{\mathbb{R}^N}\, dz$$

$$
\begin{aligned}
&+ \int_\Omega \left(\xi(z) + \mu + \frac{\|N_f(\bar{u})\|_\infty}{\bar{m}} \right) \bar{u} h \, dz + \int_{\partial\Omega} \beta(z) \bar{u} h \, d\sigma \\
&\geq \int_\Omega (1 + f(z, \bar{u})) h \, dz + \int_{\partial\Omega} \beta(z) \bar{u} h \, d\sigma \qquad \text{(see (8))} \\
&\geq \int_\Omega f(z, \bar{u}) h \, dz
\end{aligned} \tag{10}
$$

(see hypothesis $H(\beta)$ and recall $h \geq 0$).

Using $\bar{u} \in \operatorname{int} C_+$, we consider the following truncation of the reaction term $f(z, \cdot)$:

$$
\hat{f}(z, x) = \begin{cases} f(z, x), & \text{if } x \leq \bar{u}(z), \\ f(z, \bar{u}(z)), & \text{if } \bar{u}(z) < x, \end{cases} \qquad \text{for all } (z, x) \in \Omega \times \mathbb{R}. \tag{11}
$$

This is a Carathéodory function. We set $\hat{F}(z, x) = \int_0^x \hat{f}(z, x) ds$ and consider the C^1-functional $\hat{\varphi}$: $H^1(\Omega) \to \mathbb{R}$ defined by

$$
\hat{\varphi}(u) = \frac{1}{2} \vartheta(u) + \frac{\lambda_0}{2} \|u\|_2^2 - \int_\Omega \hat{F}(z, u) \, dz, \quad \text{for all } u \in H^1(\Omega).
$$

From (1), (11) and hypothesis $H(\beta)$, we see that $\hat{\varphi}$ is coercive. Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that $\hat{\varphi}$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in H^1(\Omega)$ such that

$$
\hat{\varphi}(u_0) = \inf[\hat{\varphi}(u) : u \in H^1(\Omega)]. \tag{12}
$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0, 1)$ small such that $tu(z) \leq \min\{\delta_0, \bar{m}\}$ (here $\delta_0 > 0$ is as in hypothesis $H(f)$(iv) and $0 < \bar{m} = \min_{\overline{\Omega}} \bar{u}$). Using (11) and hypothesis $H(f)$(iv), we have

$$
\begin{aligned}
\hat{\varphi}(tu) \leq \frac{t^2}{2} \|Du\|_2^2 &+ \frac{t^2}{2} \left[\|\xi^+\|_\infty + \lambda_0 \right] \|u\|_2^2 \\
&+ \frac{t^2}{2} \int_{\partial\Omega} \beta(z) u^2 \, d\sigma - \frac{\hat{c} t^d}{d} \|u\|_d^d.
\end{aligned} \tag{13}
$$

Since $d < 2$ (see hypothesis $H(f)$(iv)), by choosing $t \in (0, 1)$ even smaller if necessary, from (13) we see that

$$
\hat{\varphi}(tu) < 0 = \hat{\varphi}(0), \implies \hat{\varphi}(u_0) < 0 = \hat{\varphi}(0) \quad \text{(see (12))}, \quad \text{hence } u_0 \neq 0.
$$

From (12), we have for all $h \in H^1(\Omega)$

$$\hat{\varphi}'(u_0) = 0, \implies \langle A(u_0), h\rangle + \int_\Omega (\xi(z) + \lambda_0)u_0 h\, dz + \int_{\partial\Omega} \beta(z)u_0 h\, d\sigma = \int_\Omega \hat{f}(z, u_0)h\, dz. \quad (14)$$

In (14), first we choose $h = -u_0^- \in H^1(\Omega)$. Then

$$\begin{aligned} & \vartheta(u_0^-) + \mu\|u_0^-\|_2^2 \le 0 \quad \text{(see (11) and recall } \mu \le \lambda_0), \\ \implies & c_0\|u_0^-\|^2 \le 0 \quad \text{(see (1))}, \\ \implies & u_0 \ge 0, \quad u_0 \ne 0. \end{aligned}$$

Also, in (14) we choose $h = (u_0 - \bar{u})^+ \in H^1(\Omega)$. Using (11), we have

$$\begin{aligned} & \langle A(u_0), (u_0 - \bar{u})^+\rangle + \int_\Omega (\xi(z) + \lambda_0)u_0(u_0 - \bar{u})^+\, dz + \int_{\partial\Omega} \beta(z)u_0(u_0 - \bar{u})^+\, d\sigma \\ & \quad = \int_\Omega f(z, \bar{u})(u_0 - \bar{u})^+\, dz \\ & \quad \le \langle A(\bar{u}), (u_0 - \bar{u})^+\rangle + \int_\Omega (\xi(z) + \lambda_0)\bar{u}(u_0 - \bar{u})^+\, dz \\ & \qquad + \int_{\partial\Omega} \beta(z)\bar{u}(u_0 - \bar{u})^+\, d\sigma \quad \text{(see (10))}, \\ \implies & \langle A(u_0 - \bar{u}), (u_0 - \bar{u})^+\rangle + \int_\Omega (\xi(z) + \lambda_0)((u_0 - \bar{u})^+)^2\, dz \\ & \qquad + \int_{\partial\Omega} \beta(z)((u_0 - \bar{u})^+)^2\, d\sigma \le 0, \\ \implies & c_0\|(u_0 - \bar{u})^+\|^2 \le 0 \quad \text{(see (1) and hypothesis } H(\beta)), \\ \implies & u_0 \le \bar{u}. \end{aligned}$$

Therefore, we have proved that

$$u_0 \in [0, \bar{u}] = \{u \in W^{1,p}(\Omega) : 0 \le u(z) \le \bar{u}(z) \text{ for almost all } z \in \Omega\}, \quad u_0 \ne 0. \quad (15)$$

Then (11) and (15), imply that equation (14) becomes

$$\langle A(u_0), h\rangle + \int_\Omega (\xi(z) + \lambda_0)u_0 h\, dz + \int_{\partial\Omega} \beta(z)u_0 h\, d\sigma = \int_\Omega f(z, u_0)h\, dz, \quad \text{for all } h \in H^1(\Omega),$$

$$\implies u_0 \in S(\lambda_0) \subseteq \operatorname{int} C_+ \quad \text{and so } \lambda_0 \in \mathscr{L} \neq \emptyset$$

(see Papageorgiou and Rădulescu [12] and Proposition 3.3).

Now let $\lambda \in \mathscr{L}$ and $\eta > \lambda$. We can find $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 3.3) and we have

$$-\Delta u_\lambda(z) + (\xi(z) + \eta)u_\lambda(z) \geq -\Delta u_\lambda(z) + (\xi(z) + \lambda)u_\lambda(z), \quad \text{for almost all } z \in \Omega. \tag{16}$$

Then we truncate $f(z, \cdot)$ from above at $u_\lambda(z)$ (see (11) with $\bar{u}(z)$ replaced by $u_\lambda(z)$). Reasoning as before with λ_0 replaced by λ and using this time (16) instead of (10), via the direct method of the calculus of variations, we produce

$$u_\eta \in [0, u_\lambda] \cap S(\eta) \subseteq [0, u_\lambda] \cap \operatorname{int} C_+, \implies \eta \in \mathscr{L} \quad \text{and so } [\lambda, +\infty) \subseteq \mathscr{L}.$$

REMARK 3.5. A careful reading of the above proof, reveals that in fact we have established the following useful monotonicity property for the positive solutions of problem (P_λ) as the parameter $\lambda > 0$ varies:

"If $\lambda \in \mathscr{L}$, $u_\lambda \in S(\lambda)$ and $\eta > \lambda$, then we can find $u_\eta \in S(\eta) \subseteq \operatorname{int} C_+$ such that $u_\eta \leq u_\lambda$".

In fact we can improve this monotonicity property, provided that we strengthen a little the conditions on the reaction term $f(z, \cdot)$.

So, the new hypotheses on $f(z, x)$ are the following:

$H(f)'$: $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$, for almost all $z \in \Omega$, hypotheses $H(f)'$(i), (ii), (iii) and (iv) are the same as the corresponding hypotheses $H(f)$(i), (ii), (iii) and (iv), and

(v) for every $0 < \vartheta < \nu$, there exists $\hat{\xi}_{\vartheta,\nu} > 0$ such that for almost all $z \in \Omega$ the mapping $x \mapsto f(z, x) + \hat{\xi}_{\vartheta,\nu} x$ is nondecreasing on $[\vartheta, \nu]$.

REMARK 3.6. If $f(z, \cdot)$ is differentiable on $\mathbb{R}$, for almost all $z \in \Omega$, and for every $0 < \vartheta < \nu$, there exists $\hat{a}_{\vartheta,\nu} \in L^\infty(\Omega)_+$ such that $|f'_x(z, x)| \leq \hat{a}_{\vartheta,\nu}(z)$, for almost all $z \in \Omega$, all $x \in [\vartheta, \nu]$, then hypothesis $H(f)'$(v) is satisfied. The two examples given in the Remarks after hypotheses $H(f)$, both satisfy the new hypotheses.

PROPOSITION 3.7. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and $H(f)'$ hold, $\lambda \in \mathscr{L}$, $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ and $\eta > \lambda$. Then we can find $u_\eta \in S(\eta) \subseteq \operatorname{int} C_+$ such that $u_\lambda - u_\eta \in \operatorname{int} C_+$.*

PROOF. As we already remarked, from Proposition 3.4 and its proof, we have

"$\eta \in \mathscr{L}$ and we can find $u_\eta \in S(\eta) \subseteq \operatorname{int} C_+$ such that $u_\eta \le u_\lambda$, $u_\eta \neq u_\lambda$".

Let $\vartheta = \min_{\overline{\Omega}} u_\eta > 0$ and $\nu = \|u_\lambda\|_\infty > 0$. According to hypothesis $H(f)'$(v), we can find $\hat{\xi}_{\vartheta,\nu} > 0$ such that, for almost all $z \in \Omega$, the function $x \mapsto f(z,x) + \hat{\xi}_{\vartheta,\nu} x$ is nondecreasing on $[\vartheta, \nu]$. We have

$$
\begin{aligned}
&-\Delta u_\eta(z) + (\xi(z) + \lambda + \hat{\xi}_{\vartheta,\nu}) u_\eta(z) \\
&\quad = -\Delta u_\eta(z) + (\xi(z) + \eta + \hat{\xi}_{\vartheta,\nu}) u_\eta(z) - (\eta - \lambda) u_\eta(z) \\
&\quad \le f(z, u_\eta(z)) + \hat{\xi}_{\vartheta,\nu} u_\eta(z) \quad (\text{since } u_\eta \in \operatorname{int} C_+ \text{ and } \lambda < \eta) \\
&\quad \le f(z, u_\lambda(z)) + \hat{\xi}_{\vartheta,\nu} u_\lambda(z) \quad (\text{see hypothesis } H(f)'(\text{v}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and recall that } u_\eta \le u_\lambda) \\
&\quad = -\Delta u_\lambda(z) + (\xi(z) + \lambda + \hat{\xi}_{\vartheta,\nu}) u_\lambda(z) \text{ for almost all } z \in \Omega, \\
&\Longrightarrow \Delta(u_\lambda - u_\eta)(z) \\
&\qquad \le (\xi(z) + \lambda + \hat{\xi}_{\vartheta,\nu})(u_\lambda - u_\eta)(z) \\
&\qquad \le (\xi^+(z) + \lambda + \hat{\xi}_{\vartheta,\nu})(u_\lambda - u_\eta)(z) \\
&\qquad \le (\left\|\xi^+\right\|_\infty + \lambda + \hat{\xi}_{\vartheta,\nu})(u_\lambda - u_\eta)(z), \quad \text{for almost all } z \in \Omega \\
&\qquad (\text{see hypothesis } H(\xi)), \\
&\Longrightarrow u_\lambda - u_\eta \in \operatorname{int} C_+ \quad (\text{by the strong maximum principle}).
\end{aligned}
$$

In what follows, for each $\lambda > 0$, we denote by $\varphi_\lambda \colon H^1(\Omega) \to \mathbb{R}$ the energy functional for problem (P_λ) defined by

$$
\varphi_\lambda(u) = \frac{1}{2}\vartheta(u) + \frac{\lambda}{2}\|u\|_2^2 - \int_\Omega F(z,u)\,dz, \quad \text{for all } u \in H^1(\Omega).
$$

We know that $\varphi_\lambda \in C^1(H^1(\Omega))$.

Let $\lambda_* = \inf \mathscr{L} \ge 0$.

PROPOSITION 3.8. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then $\lambda_* > 0$.*

PROOF. We argue indirectly. So, suppose we can find $\{\lambda_n\}_{n\ge1} \subseteq \mathscr{L}$ such that $\lambda_n \downarrow 0$. From the last part of the proof of Proposition 3.4, we know that we can find a nondecreasing sequence $\{u_n\}_{n\ge1} \subseteq H^1(\Omega)$ such that $u_n \in S(\lambda_n) \subseteq \operatorname{int} C_+$ and

$$
\varphi_{\lambda_n}(u_n) < 0, \quad \text{for all } n \in \mathbb{N}. \tag{17}
$$

From (17), we have

$$-\int_\Omega 2F(z,u_n),dz \le -\|Du_n\|_2^2 - \int_\Omega (\xi(z)+\lambda_n)u_n^2\,dz - \int_{\partial\Omega}\beta(z)u_n^2\,d\sigma, \quad \text{for all } n\in\mathbb{N}. \tag{18}$$

On the other hand, since $u_n \in S(\lambda_n)$ for all $n\in\mathbb{N}$, we have

$$\langle A(u_n),h\rangle + \int_\Omega (\xi(z)+\lambda_n)u_n h\,dz + \int_{\partial\Omega}\beta(z)u_n h\,d\sigma = \int_\Omega f(z,u_n)h\,dz, \quad \text{for all } h\in H^1(\Omega). \tag{19}$$

In (19) we choose $h=u_n\in H^1(\Omega)$. Then

$$\int_\Omega f(z,u_n)u_n\,dz = \|Du_n\|_2^2 + \int_\Omega (\xi(z)+\lambda_n)u_n^2\,dz + \int_{\partial\Omega}\beta(z)u_n^2 d\sigma, \quad \text{for all } n\in\mathbb{N}. \tag{20}$$

We add (18) and (20) and obtain

$$\int_\Omega [f(z,u_n)u_n - 2F(z,u_n)]dz \le 0, \quad \text{for all } n\in\mathbb{N}. \tag{21}$$

Hypotheses $H(f)$(i) and (iii) imply that we can find $\gamma_1\in(0,\gamma_0)$ and $c_1>0$ such that

$$\gamma_1 x^\tau - c_1 \le f(z,x)x - 2F(z,x), \quad \text{for almost all } z\in\Omega, \text{ all } x\ge 0. \tag{22}$$

Using (22) in (21), we obtain

$$\{u_n\}_{n\ge1}\subseteq L^\tau(\Omega) \quad \text{is bounded.} \tag{23}$$

First suppose $N\neq 2$. From hypothesis $H(f)$(iii) it is clear that without any loss of generality, we may assume that $\tau<r<2^*$. Let $t\in(0,1)$ be such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{2^*} \tag{24}$$

(recall that if $N=1$, then $2^*=+\infty$). Using the interpolation inequality (see, for example, Gasinski and Papageorgiou [4, p. 905]), we have

$$\|u_n\|_r \le \|u_n\|_\tau^{1-t}\,\|u_n\|_{2^*}^t \implies \|u_n\|_r^r \le c_2\,\|u_n\|^{tr}, \quad \text{for some } c_2>0, \text{ all } n\in\mathbb{N} \quad (\text{see (23)}). \tag{25}$$

We can always assume that $r \geq 2N/(N+1)$ (see hypothesis $H(f)$(i)). Then we have $\frac{2}{r} + \frac{1}{N} \leq 1$ and so $\frac{2}{r} + \frac{1}{s} < 1$ (see hypothesis $H(\xi)$). We have $u_n^2 \in L^{r/2}(\Omega)$, hence by the generalized Hölder inequality (see, for example, Gasinski and Papageorgiou [4, p. 904]), we have

$$\left|\int_\Omega \xi(z) u_n^2 dz\right| \leq \|\xi\|_s \, \|u_n\|_r^2 \leq c_3(1 + \|u_n\|_r^r),$$
$$\text{for some } c_3 > 0, \text{ all } n \in \mathbb{N} \quad (\text{recall that } 2 < r). \tag{26}$$

Also, using hypothesis $H(f)$(i), we see that

$$\left|\int_\Omega f(z, u_n) u_n dz\right| \leq c_4(1 + \|u_n\|_r^r), \quad \text{for some } c_4 > 0, \text{ all } n \in \mathbb{N}. \tag{27}$$

Returning to (20) and using (26), (27) and hypothesis $H(\beta)$, we have

$$\begin{aligned}
\|Du_n\|_2^2 &\leq c_5(1 + \|u_n\|_r^r), \quad \text{for some } c_5 > 0, \text{ all } n \in \mathbb{N} \\
&\leq c_6(1 + \|u_n\|^{tr}), \quad \text{for some } c_6 > 0, \text{ all } n \in \mathbb{N} \text{ (see (25))}, \\
\implies \quad & \|Du_n\|_2^2 + \|u_n\|_\tau^2 \leq c_7(1 + \|u_n\|^{tr}), \\
& \text{for some } c_7 > 0, \text{ all } n \in \mathbb{N} \text{ (see (23))}.
\end{aligned} \tag{28}$$

Recall that $u \mapsto \|Du\|_2 + \|u\|_\tau$ is an equivalent norm on $H^1(\Omega)$ (see, for example, Gasinski and Papageorgiou [4, p. 227]). Then, from (28), we have

$$\|u_n\|^2 \leq c_8(1 + \|u_n\|^{tr}), \quad \text{for some } c_8 > 0, \text{ all } n \in \mathbb{N}. \tag{29}$$

The restriction on τ (see hypothesis $H(f)$(iii)) and (24) imply that $tr < 2$ and so we infer that

$$\{u_n\}_{n \geq 1} \subseteq H^1(\Omega) \quad \text{is bounded}. \tag{30}$$

If $N = 2$, then $2^* = +\infty$ and from the Sobolev embedding theorem, we have $H^1(\Omega) \hookrightarrow L^\eta(\Omega)$, for every $\eta \in [1, +\infty)$. So, the above argument works if we replace 2^* by $q > 1$ large enough such that $\max\{r-2, 1\} < \tau < r < q$ and

$$tr = q\frac{r-\tau}{q-\tau} < 2 \quad \left(\text{note that } q\frac{r-\tau}{q-\tau} \to r - \tau < 2 \text{ as } q \to 2^* = +\infty\right).$$

Then again we reach (30).

Therefore, we always have (30) and so we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } H^1(\Omega) \quad \text{and} \quad u_n \to u_*$$
$$\text{in } L^r(\Omega) \text{ and in } L^2(\partial\Omega), \quad u_* \geq 0. \tag{31}$$

If in (19) we pass to the limit as $n \to \infty$ and use (31), then

$$\langle A(u_*), h\rangle + \int_\Omega \xi(z)u_* h\,dz + \int_{\partial\Omega} \beta(z)u_* h\,d\sigma \\ = \int_\Omega f(z,u_*)h\,dz, \quad \text{for all } h \in H^1(\Omega). \tag{32}$$

Also, we have

$$u_1 \leq u_n \text{ for all } n \in \mathbb{N} \implies u_1 \leq u_* \text{ and so } u_* \neq 0. \tag{33}$$

From (32) we have

$$-\Delta u_*(z) + \xi(z)u_*(z) = f(z,u_*(z)), \quad \text{for almost all } z \in \Omega,$$
$$\frac{\partial u_*}{\partial n} + \beta(z)u_* = 0, \quad \text{on } \partial\Omega \quad \text{(see Papageorgiou and Rădulescu [12])},$$
$$\implies u_* \in \operatorname{int} C_+ \quad \text{(using the regularity result of Wang [17] and (33))}.$$

In (32) we choose $h = \hat{u}_1 \in \operatorname{int} C_+$. Then

$$\hat{\lambda}_1 \int_\Omega \hat{u}_1 u_*\,dz = \int_\Omega f(z,u_*)\hat{u}_1\,dz \geq \int_\Omega \eta(z)\hat{u}_1 u_*\,dz \\ \text{(see hypothesis } H(f)\text{(iv))},$$
$$\implies \int_\Omega (\hat{\lambda}_1 - \eta(z))\hat{u}_1 u_*\,dz \geq 0. \tag{34}$$

But $(\hat{u}_1 u_*)(z) > 0$, for all $z \in \overline{\Omega}$ (recall $\hat{u_1}, u_* \in \operatorname{int} C_+$), and $\hat{\lambda}_1 - \eta(z) \leq 0$, for almost all $z \in \Omega$, with strict inequality on a set of positive measure. Therefore

$$\int_\Omega (\hat{\lambda}_1 - \eta(z))\hat{u}_1 u_*\,dz < 0. \tag{35}$$

Comparing (34) and (35), we reach a contradiction. This proves that $\lambda_* > 0$.

PROPOSITION 3.9. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and $H(f)'$ hold and $\lambda \in (\lambda_*, +\infty)$. Then problem (P_λ) has at least two positive solutions*

$$u_\lambda, \hat{u}_\lambda \in \operatorname{int} C_+, \quad u_\lambda \neq \hat{u}_\lambda.$$

PROOF. Let $\eta_1, \eta_2 \in \mathscr{L}$ with $\eta_1 < \lambda < \eta_2$. From Proposition 3.7 we know that we can find $u_{\eta_1} \in S(\eta_1) \subseteq \operatorname{int} C_+$ and $u_{\eta_2} \in S(\eta_2) \subseteq \operatorname{int} C_+$ such that

$u_{\eta_1} - u_{\eta_2} \in \operatorname{int} C_+$. Using these two solutions, we introduce the following truncation-perturbation of $f(z, \cdot)$:

$$k(z,x) = \begin{cases} f(z, u_{\eta_2}(z)) + \mu u_{\eta_2}(z), & \text{if } x < u_{\eta_2}(z), \\ f(z,x) + \mu x, & \text{if } u_{\eta_2}(z) \le x \le u_{\eta_1}(z), \\ f(z, u_{\eta_1}(z)) + \mu u_{\eta_1}(z), & \text{if } u_{\eta_1}(z) < x. \end{cases} \tag{36}$$

Here $\mu > 0$ is as in (1). Clearly $k(z,x)$ is a Carathéodory function. We set $K(z,x) = \int_0^x k(z,s)\,ds$ and consider the C^1-functional $\psi_\lambda \colon H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{2}\vartheta(u) + \frac{\lambda+\mu}{2}\,\|u\|_2^2 - \int_\Omega K(z,u)\,dz, \quad \text{for all } u \in H^1(\Omega).$$

From (1) and (35) it is clear that ψ_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\lambda \in H^1(\Omega)$ such that

$$\begin{aligned} &\psi_\lambda(u_\lambda) = \inf[\psi_\lambda(u) : u \in H^1(\Omega)], \\ \Longrightarrow\ &\psi'_\psi(u_\lambda) = 0, \\ \Longrightarrow\ &\langle A(u_\lambda), h\rangle + \int_\Omega (\xi(z) + \lambda + \mu) u_\lambda h\,dz + \int_{\partial\Omega} \beta(z) u_\lambda h\,d\sigma \\ &\qquad = \int_\Omega k(z, u_\lambda) h\,dz, \quad \text{for all } h \in H^1(\Omega). \end{aligned} \tag{37}$$

In (37) first we choose $h = (u_\lambda - u_{\eta_1})^+ \in H^1(\Omega)$. We have

$$\begin{aligned} &\langle A(u_\lambda), (u_\lambda - u_{\eta_1})^+\rangle + \int_\Omega (\xi(z) + \lambda + \mu) u_\lambda (u_\lambda - u_{\eta_1})^+\,dz \\ &\qquad + \int_{\partial\Omega} \beta(z) u_\lambda (u_\lambda - u_{\eta_1})^+\,d\sigma \\ &= \int_\Omega [f(z, u_{\eta_1}) + \mu u_{\eta_1}](u_\lambda - u_{\eta_1})^+\,dz \quad \text{(see (36))} \\ &= \langle A(u_{\eta_1}), (u_\lambda - u_{\eta_1})^+\rangle + \int_\Omega (\xi(z) + \eta_1 + \mu) u_{\eta_1} (u_\lambda - u_{\eta_1})^+\,dz \\ &\qquad + \int_{\partial\Omega} \beta(z) u_{\eta_1} (u_\lambda - u_{\eta_1})^+\,d\sigma \quad (\text{since } u_{\eta_1} \in S(\eta_1)) \\ &\le \langle A(u_{\eta_1}), (u_\lambda - u_{\eta_1})^+\rangle + \int_\Omega (\xi(z) + \lambda + \mu) u_{\eta_1} (u_\lambda - u_{\eta_1})^+\,dz \\ &\qquad + \int_{\partial\Omega} \beta(z) u_{\eta_1} (u_\lambda - u_{\eta_1})^+\,d\sigma \ (\text{since } \eta_1 < \lambda,\ u_{\eta_1} \in \operatorname{int} C_+) \end{aligned}$$

$$\begin{aligned}\Longrightarrow\quad & \|D(u_\lambda - u_{\eta_1})^+\|_2^2 + \int_\Omega (\xi(z)+\mu)((u_\lambda - u_{\eta_1})^+)^2\,dz \\ &\qquad\qquad + \int_{\partial\Omega} \beta(z)((u_\lambda - u_{\eta_1})^+)^2\,d\sigma \le 0,\\ \Longrightarrow\quad & c_0\|(u_\lambda - u_{\eta_1})^+\|^2 \le 0 \quad \text{(see (1))},\\ \Longrightarrow\quad & u_\lambda \le u_{\eta_1}.\end{aligned}$$

Similarly, if in (37) we choose $h = (u_{\eta_2} - u_\lambda)^+ \in H^1(\Omega)$, then we obtain

$$u_{\eta_2} \le u_\lambda.$$

So, we have proved that

$$\begin{aligned} u_\lambda &\in [u_{\eta_2}, u_{\eta_1}]\\ &= \{u \in H^1(\Omega) : u_{\eta_2}(z) \le u(z) \le u_{\eta_1}(z) \text{ for almost all } z \in \Omega\}.\end{aligned}$$

In fact, as in the proof of Proposition 3.7, using the strong maximum principle (see, for example, Gasinski and Papageorgiou [4, p. 738]), we have

$$u_\lambda - u_{\eta_2} \in \operatorname{int} C_+ \quad \text{and} \quad u_{\eta_1} - u_\lambda \in \operatorname{int} C_+ \\ \Longrightarrow \quad u_\lambda \in \operatorname{int}_{C^1(\overline{\Omega})}[u_{\eta_2}, u_{\eta_1}]. \tag{38}$$

Because of (38) and (36), equation (37) becomes

$$\langle A(u_\lambda), h\rangle + \int_\Omega (\xi(z)+\lambda)u_\lambda h\,dz + \int_{\partial\Omega} \beta(z)u_\lambda h\,d\sigma = \int_\Omega f(z,u_\lambda)h\,dz \\ \text{for all } h \in H^1(\Omega),\\ \Longrightarrow \quad u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$$

(see Papageorgiou and Rădulescu [12] and Proposition 3.3).

Next we consider the following truncation-perturbation of the reaction $f(z,\cdot)$:

$$g(z,x) = \begin{cases} f(z, u_{\eta_2}(z)) + \mu u_{\eta_2}(z), & \text{if } x \le u_{\eta_2}(z),\\ f(z,x) + \mu x, & \text{if } u_{\eta_2}(z) < x.\end{cases} \tag{39}$$

This is a Carathéodory function. We set $G(z,x) = \int_0^x g(z,s)\,ds$ and consider the C^1-functional $\hat{\varphi}_\lambda\colon H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda(u) = \frac{1}{2}\vartheta(u) + \frac{\lambda+\mu}{2}\|u\|_2^2 - \int_\Omega G(z,u)\,dz, \quad \text{for all } u \in H^1(\Omega).$$

Let $u \in K_{\hat{\psi}_\lambda}$. Then for all $h \in H^1(\Omega)$

$$\hat{\psi}_\lambda'(u) = 0 \implies \langle A(u), h\rangle + \int_\Omega (\xi(z) + \lambda + \mu)uh\,dz \\ + \int_{\partial\Omega} \beta(z)uh\,d\sigma = \int_\Omega g(z,u)h\,dz. \quad (40)$$

In (40), we choose $h = (u_{\eta_2} - u)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
&\langle A(u), (u_{\eta_2} - u)^+\rangle + \int_\Omega (\xi(z) + \lambda + \mu)u(u_{\eta_2} - u)^+\,dz \\
&\qquad\qquad + \int_{\partial\Omega} \beta(z)u(u_{\eta_2} - u)^+\,d\sigma \\
&\quad = \int_\Omega [f(z,u_{\eta_2}) + \mu u_{\eta_2}](u_{\eta_2} - u)^+\,dz \quad \text{(see (39))} \\
&\quad = \langle A(u_{\eta_2}), (u_{\eta_2} - u)^+\rangle + \int_\Omega (\xi(z) + \eta_2 + \mu)u_{\eta_2}(u_{\eta_2} - u)^+\,dz \\
&\qquad\qquad + \int_{\partial\Omega} \beta(z)u_{\eta_2}(u_{\eta_2} - u)^+\,d\sigma \quad (\text{since } u_{\eta_2} \in S(\eta_2)) \\
&\quad \geq \langle A(u_{\eta_2}), (u_{\eta_2} - u)^+\rangle + \int_\Omega (\xi(z) + \lambda + \mu)u_{\eta_2}(u_{\eta_2} - u)^+\,dz \\
&\qquad\qquad + \int_{\partial\Omega} \beta(z)u_{\eta_2}(u_{\eta_2} - u)^+\,d\sigma \quad (\text{since } \lambda < \eta_2,\ u_{\eta_2} \in \operatorname{int} C_+), \\
&\implies \left\| D(u_{\eta_2} - u)^+\right\|_2^2 + \int_\Omega (\xi(z) + \lambda + \mu)(u_{\eta_2} - u)^+\,dz \\
&\qquad\qquad + \int_{\partial\Omega} \beta(z)((u_{\eta_2} - u)^+)^2\,d\sigma \leq 0, \\
&\implies c_0 \left\|(u_{\eta_2} - u)^+\right\|^2 \leq 0 \quad \text{(see (1))}, \\
&\implies u_{\eta_2} \leq u, \\
&\implies K_{\hat{\psi}_\lambda} \subseteq [u_{\eta_2}) = \{u \in H^1(\Omega) : u_{\eta_2}(z) \leq u(z) \\
&\qquad\qquad \text{for almost all } z \in \Omega\}. \quad (41)
\end{aligned}$$

Then relations (41) and (39) imply that

$$K_{\hat{\psi}_\lambda} \subseteq S(\lambda) \subseteq \operatorname{int} C_+. \quad (42)$$

From (36) and (39) we see that

$$\hat{\psi}_\lambda|_{[u_{\eta_2}, u_{\eta_1}]} = \psi_\lambda|_{[u_{\eta_2}, u_{\eta_1}]}. \quad (43)$$

Recall that $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ is a minimizer of ψ_λ. Then from (43) and (38) it follows that

$$\begin{aligned} & u_\lambda \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \hat{\psi}_\lambda, \\ \Longrightarrow\ & u_\lambda \text{ is a local } H^1(\Omega)\text{-minimizer of } \hat{\psi}_\lambda \end{aligned}$$

(see Proposition 2.2).

We assume that $K_{\hat{\psi}_\lambda}$ is finite (otherwise from (42) we see that we already have an infinity of distinct positive solutions of problem (P_λ) and so we are done). Hence we can find $\rho \in (0, 1)$ small such that

$$\hat{\psi}_\lambda(u_\lambda) < \inf[\hat{\psi}_\lambda(u) : \|u - u_\lambda\| = \rho] = \hat{m}_\lambda \tag{44}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29). Hypothesis $H(f)'$(ii) implies that for any $u \in \operatorname{int} C_+$, we have

$$\hat{\psi}_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{45}$$

Moreover, reasoning as in the proof of Proposition 3.8, we can show that every Cerami sequence of the functional $\hat{\psi}_\lambda$ is bounded. From this and the fact that $H^1(\Omega)$ being a Hilbert space it satisfies the Kadec-Klee property, we infer that

$$\hat{\psi}_\lambda \text{ satisfies the } C\text{-condition.} \tag{46}$$

Then (44), (45) and (46) permit the use of Theorem 2.1 (the mountain pass theorem) and so we can find $\hat{u}_\lambda \in H^1(\Omega)$ such that

$$\hat{u}_\lambda \in K_{\hat{\psi}_\lambda} \subseteq S(\lambda) \subseteq \operatorname{int} C_+ \text{ (see (42)) } \quad \text{and} \quad \hat{m}_\lambda \leq \hat{\psi}_\lambda(\hat{u}_\lambda). \tag{47}$$

From (47) and (44) we see that $u_\lambda \neq \hat{u}_\lambda$.

Next we examine what happens in the critical case $\lambda = \lambda_*$.

PROPOSITION 3.10. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then $\lambda_* \in \mathscr{L}$.*

PROOF. Let $\{\lambda_n\}_{n\leq 1} \subseteq (\lambda_*, +\infty)$ be such that $\lambda_n \downarrow \lambda_*$. We can find a nondecreasing sequence $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that $u_n \in S(\lambda_n) \subseteq \operatorname{int} C_+$ and

$$\varphi_{\lambda_n}(u_n) < 0, \quad \text{for all } n \in \mathbb{N}.$$

As in the proof of Proposition 3.7, we show that $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ is bounded. Hence we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } H^1(\Omega) \quad \text{and} \quad u_n \to u_* \text{ in } L^r(\Omega) \text{ and in } L^2(\partial\Omega). \tag{48}$$

We have

$$\langle A(u_n), h\rangle + \int_\Omega (\xi(z)+\lambda_n)u_n h\,dz + \int_{\partial\Omega} \beta(z)u_n h\,d\sigma \\ = \int_\Omega f(z,u_n)h\,dz, \quad \text{for all } h \in H^1(\Omega). \tag{49}$$

Passing to the limit as $n \to \infty$ in (49) and using (48), we obtain

$$\langle A(u_*), h\rangle + \int_\Omega (\xi(z)+\lambda_*)u_* h\,dz + \int_{\partial\Omega} \beta(z)u_* h\,d\sigma \\ = \int_\Omega f(z,u_*)h\,dz, \quad \text{for all } h \in H^1(\Omega). \tag{50}$$

Also we have $u_1 \le u_n$ for all $n \in \mathbb{N}$, hence

$$\begin{aligned} & u_1 \le u_*, \\ \Longrightarrow\ & u_* \in S(\lambda_*) \subseteq \operatorname{int} C_+ \quad \text{(see (50))}, \\ \Longrightarrow\ & \lambda_* \in \mathscr{L}. \end{aligned}$$

COROLLARY 3.11. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then* $\mathscr{L} = [\lambda_*, +\infty)$.

Next we show that for every $\lambda \in \mathscr{L}$, problem (P_λ) admits a smallest positive solution $u_\lambda^* \in S(\lambda) \subseteq \operatorname{int} C_+$ and then we establish the monotonicity and continuity properties of the map $\lambda \mapsto u_\lambda^*$.

By hypothesis $H(\xi)$, if $N \ge 2$, then $s > N$ and so $s' < N' = N/(N-1) < 2^*$, while the case $N = 1$ is easy because then $W^{1,2}(0,b) \hookrightarrow C[0,b]$. From hypotheses $H(f)$(i) and (iv), we see that for $\eta > \max\{r, s'\}$ we can find $c_9 > 0$ such that

$$f(z,x) \ge \hat{c}x^{d-1} - c_9 x^{\eta-1}, \quad \text{for almost all } z \in \Omega,\ \text{all } x \ge 0. \tag{51}$$

For $\chi \ge 0$, we consider the following auxiliary Robin problem

$$\left\{ \begin{aligned} & -\Delta u(z) + (\xi(z)+\chi)u(z) = \hat{c}u(z)^{d-1} - c_9 u(z)^{r-1}, && \text{in } \Omega, \\ & \frac{\partial u}{\partial n} + \beta(z)u = 0, && \text{on } \partial\Omega,\ u > 0. \end{aligned} \right\} \tag{52}$$

PROPOSITION 3.12. *If hypotheses $H(\xi)$ and $H(\beta)$ hold and $\chi \ge 0$, then problem* (52) *admits a unique positive solution $\tilde{u} \in \operatorname{int} C_+$.*

PROOF. First we establish the existence of a positive solution. To this end, let $\tilde{\sigma}\colon H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\begin{aligned}\tilde{\sigma}(u) &= \frac{1}{2}\vartheta(u) + \frac{\mu}{2}\|u^-\|_2^2 + \frac{c_9}{\eta}\|u^+\|_\eta^\eta - \frac{\hat{c}}{d}\|u^+\|_d^d + \frac{\chi}{2}\|u\|_2^2 \\ &\geq \frac{c_0}{2}\|u^-\|^2 + \frac{1}{2}\vartheta(u^+) + \frac{c_9}{\eta}\|u^+\|_\eta^\eta - \frac{\hat{c}}{d}\|u^+\|_d^d,\end{aligned} \tag{53}$$

for all $u \in H^1(\Omega)$ (see (1)). Note that $(u^+)^2 \in L^{2^*/2}(\Omega) \subseteq L^{s'/2}(\Omega)$. So, using Hölder's inequality, we have

$$\int_\Omega \xi(z)(u^+)^2\,dz \leq \|\xi\|_s\|u^+\|_{s'}^2 \quad \text{(see hypothesis } H(\xi)). \tag{54}$$

Using (54) in (53), we obtain

$$\begin{aligned}\tilde{\sigma}(u) &\geq \frac{c_0}{2}\|u^-\|^2 + \frac{1}{2}\|Du^+\|_2^2 + \frac{c_9}{\eta}\|u^+\|_\eta^\eta - c_{10}\|u^+\|_\eta^2 - \frac{\hat{c}}{d}\|u^+\|_d^d \\ &\quad \text{for some } c_{10} > 0 \quad \text{(see hypothesis } H(\beta) \text{ and recall that } \eta > s') \\ &= \frac{c_0}{2}\|u^-\|^2 + \frac{1}{2}\|Du^+\|_2^2 + \left(\frac{c_9}{\eta}\|u^+\|_\eta^{\eta-2} - c_{10}\right)\|u^+\|_\eta^2 - \frac{\hat{c}}{d}\|u^+\|_d^d.\end{aligned} \tag{55}$$

Since $d < 2 < \eta$, from (55) it is clear that $\tilde{\sigma}(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in H^1(\Omega)$ such that

$$\tilde{\sigma}(\tilde{u}) = \inf[\tilde{\sigma}(u) : u \in H^1(\Omega)]. \tag{56}$$

As before (see the proof of Proposition 3.4), since $d < 2 < \eta$, given $u \in \operatorname{int} C_+$ and choosing $t \in (0, 1)$ appropriately small, we have

$$\begin{aligned}&\tilde{\sigma}(tu) < 0 = \tilde{\sigma}(0), \\ \Longrightarrow\ &\tilde{\sigma}(\tilde{u}) < 0 = \tilde{\sigma}(0) \quad \text{(see (56))}, \\ \Longrightarrow\ &\tilde{u} \neq 0.\end{aligned}$$

From (56), we have

$$\begin{aligned}&\tilde{\sigma}'(\tilde{u}) = 0, \\ \Longrightarrow\ &\langle A(\tilde{u}, h)\rangle + \int_\Omega (\xi(z) + \chi)\tilde{u}h\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}h\,d\sigma - \mu\int_\Omega \tilde{u}^-h\,dz \\ &\quad = \hat{c}\int_\Omega (\tilde{u}^+)^{d-1}h\,dz - c_9\int_\Omega (\tilde{u}^+)^{\eta-1}h\,dz, \quad \text{for all } h \in H^1(\Omega).\end{aligned} \tag{57}$$

In (57) we choose $h = -\tilde{u}^- \in H^1(\Omega)$. Then we obtain

$$
\begin{aligned}
& \vartheta(\tilde{u}^-) + \mu\|\tilde{u}^-\|_2^2 \leq 0 \quad \text{(recall } \chi \geq 0\text{)}, \\
\Longrightarrow\ & c_0\|\tilde{u}^-\|^2 \leq 0 \qquad\qquad \text{(see (1))}, \\
\Longrightarrow\ & \tilde{u} \geq 0, \tilde{u} \neq 0.
\end{aligned}
$$

Then (57) become

$$
\begin{aligned}
& \langle A(\tilde{u}), h\rangle + \int_\Omega (\xi(z) + \chi)\tilde{u}h\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}h\,d\sigma \\
& \qquad = \hat{c}\int_\Omega \tilde{u}^{d-1}h\,dz - c_9\int_\Omega \tilde{u}^{\eta-1}h\,dz, \quad \text{for all } h \in H^1(\Omega), \\
& \Longrightarrow \quad \tilde{u} \text{ is a positive solution of the auxiliary problem (52).}
\end{aligned}
$$

As before (see the proof or Proposition 3.3), using the regularity results of Wang [17], we have that $\tilde{u} \in C_+ \setminus \{0\}$. Moreover, we have

$$
\begin{aligned}
\Delta\tilde{u}(z) &\leq (\xi(z) + \chi + c_9\tilde{u}(z)^{\eta-2})\tilde{u}(z) \\
&\leq (\xi^+(z) + \chi + c_9\,\|\tilde{u}\|_\infty^{\eta-2})\tilde{u}(z) \\
&\leq (\left\|\xi^+\right\|_\infty + \chi + c_9\,\|\tilde{u}\|_\infty^{\eta-2})\tilde{u}(z), \quad \text{for almost all } z \in \Omega \\
&\qquad \text{(see hypothesis } H(\xi) \text{ and recall } \eta > 2\text{)}, \\
\Longrightarrow\ & \tilde{u} \in \operatorname{int} C_+ \quad \text{(by the strong maximum principle).}
\end{aligned}
$$

Next we show the uniqueness of this positive solution $\tilde{u} \in \operatorname{int} C_+$. As in Filippakis and Papageorgiou [3, Lemma 4.1] (see also Motreanu, Motreanu and Papageorgiou [8, p. 421]), we show that the set of positive solutions of the auxiliary problem (52), is downward directed (that is, if u_1, u_2 are positive solutions of (52), then we can find u a positive solution of (52) such that $u \leq u_1, u \leq u_2$). So, if $\tilde{v} \in H^1(\Omega)$ is another positive solution of (52), then as for $\tilde{u}$, we can show that $\tilde{v} \in \operatorname{int} C_+$ and without any loss of generality we may assume that $\tilde{v} \leq \tilde{u}$. We have

$$
\begin{aligned}
& \langle A(\tilde{u}), h\rangle + \int_\Omega (\xi(z) + \chi)\tilde{u}h\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}h\,d\sigma \\
& \qquad\qquad = \int_\Omega \hat{c}\tilde{u}^{d-1}h\,dz - \int_\Omega c_9\tilde{u}^{\eta-1}h\,dz
\end{aligned} \tag{58}
$$

$$\begin{aligned}\langle A(\tilde{v}), h\rangle &+ \int_\Omega (\xi(z)+\chi)\tilde{v}h\,dz + \int_{\partial\Omega}\beta(z)\tilde{v}h\,d\sigma \\ &= \int_\Omega \hat{c}\tilde{v}^{d-1}h\,dz - \int_\Omega c_9\tilde{v}^{\eta-1}h\,dz \quad \text{for all } h \in H^1(\Omega). \quad (59)\end{aligned}$$

In (58) we choose $h = \tilde{v} \in \operatorname{int} C_+$ and in (59) we choose $h = \tilde{u} \in \operatorname{int} C_+$. Then the left-hand sides of the two equations are equal. Hence, we have

$$\int_\Omega \hat{c}\left(\frac{1}{\tilde{u}^{2-d}} - \frac{1}{\tilde{v}^{2-d}}\right)\tilde{u}\tilde{v}\,dz = \int_\Omega c_9(\tilde{u}^{\eta-2} - \tilde{v}^{\eta-2})\tilde{u}\tilde{v}\,dz. \qquad (60)$$

Since $d < 2 < \eta$, we have that

$$x \mapsto \frac{1}{x^{2-d}} \text{ is strictly decreasing on } (0,+\infty)$$

and

$$x \mapsto x^{\eta-2} \text{ is strictly increasing on } (0,+\infty).$$

Then from (60) we infer that $\tilde{u} = \tilde{v}$ and this proves the uniqueness of the positive solution $\tilde{u} \in \operatorname{int} C_+$ of problem (52).

PROPOSITION 3.13. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, $\lambda \in \mathscr{L}$ and $u \in S(\lambda)$. Then $\tilde{u} \le u$ where $\tilde{u} \in \operatorname{int} C_+$ is the solution of* (52) *with $\chi \ge \lambda$.*

PROOF. Consider the Carathéodory function $e\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$e(z,x) = \begin{cases} 0, & \text{if } x < 0, \\ \hat{c}x^{d-1} - c_9x^{\eta-1} + \mu x, & \text{if } 0 \le x \le u(z), \\ \hat{c}u(z)^{d-1} - c_9u(z)^{\eta-1} + \mu u(z), & \text{if } u(z) < x. \end{cases} \qquad (61)$$

Again $\mu > 0$ is as postulated by (1). Let $E(z,x) = \int_0^x e(z,s)ds$ and for $\chi \ge \lambda$ let $\Im\colon H^1(\Omega) \to \mathbb{R}$ be defined by

$$\Im(v) = \frac{1}{2}\vartheta(v) + \frac{\mu+\chi}{2}\|v\|_2^2 - \int_\Omega E(z,v)\,dz, \quad \text{for all } v \in H^1(\Omega).$$

From (1) and (61) it is clear that $\Im(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in H^1(\Omega)$ such that

$$\Im(\tilde{u}_0) = \inf[\Im(u) : u \in H^1(\Omega)]. \qquad (62)$$

Since $d < 2 < \eta$, as before (see the proof of Proposition 3.4), given $y \in \operatorname{int} C_+$ and $t \in (0,1)$ small (at least such that $ty \le u$, recall $u \in \operatorname{int} C_+$),

we have

$$\Im(ty) < 0, \implies \Im(\tilde{u}_0) < 0 = \Im(0) \ \text{(see (62))}, \quad \text{hence } \tilde{u}_0 \neq 0.$$

From (62) we have

$$\begin{aligned} &\Im'(\tilde{u}_0) = 0, \\ \implies &\langle A(\tilde{u}_0), h\rangle + \int_\Omega (\xi(z) + \chi)\tilde{u}_0 h\, dz + \int_{\partial\Omega} \beta(z)\tilde{u}_0 h\, d\sigma \\ &\qquad + \mu \int_\Omega \tilde{u}_0 h\, dz = \int_\Omega e(z, \tilde{u}_0) h\, dz, \quad \text{for all } h \in H^1(\Omega). \end{aligned} \tag{63}$$

In (63), first we choose $h = -u_0^- \in H^1(\Omega)$. Using (61), we have

$$\begin{aligned} &\vartheta(\tilde{u}_0^-) + \mu \|\tilde{u}_0^-\|_2^2 \leq 0 \quad \text{(since } \chi \geq \lambda > 0), \\ \implies &c_0 \|\tilde{u}_0^-\|^2 \leq 0 \qquad \text{(see (1))}, \\ \implies &\tilde{u}_0 \geq 0, \tilde{u}_0 \neq 0. \end{aligned}$$

Also in (63) we choose $h = (\tilde{u}_0 - u)^+ \in H^1(\Omega)$. Then

$$\begin{aligned} &\langle A(\tilde{u}_0), (\tilde{u}_0 - u)^+\rangle + \int_\Omega (\xi(z) + \chi)\tilde{u}_0(\tilde{u}_0 - u)^+\, dz \\ &\qquad + \int_{\partial\Omega} \beta(z)\tilde{u}_0(\tilde{u}_0 - u)^+\, d\sigma + \mu \int_\Omega \tilde{u}_0(\tilde{u}_0 - u)^+\, dz \\ &\quad = \int_\Omega [\hat{c}u^{d-1} - c_9 u^{\eta-1} + \mu u](\tilde{u}_0 - u)^+\, dz \quad \text{(see (61))} \\ &\quad \leq \int_\Omega [f(z,u) + \mu u](\tilde{u}_0 - u)^+\, dz \quad \text{(see (51))} \\ &\quad = \langle A(u), (\tilde{u}_0 - u)^+\rangle + \int_\Omega (\xi(z) + \lambda) u (\tilde{u}_0 - u)^+\, dz \\ &\qquad + \int_{\partial\Omega} \beta(z) u(\tilde{u}_0 - u)^+\, d\sigma + \mu \int_\Omega u(\tilde{u}_0 - u)^+\, dz \quad \text{(since } u \in S(\lambda)) \\ \implies &\vartheta((\tilde{u}_0 - u)^+) + \mu \left\|(\tilde{u}_0 - u)^+\right\|_2^2 \leq 0 \quad \text{(since } \chi \geq \lambda), \\ \implies &c_0 \left\|(\tilde{u}_0 - u)^+\right\|^2 \leq 0 \quad \text{(see (1))}, \\ \implies &\tilde{u}_0 \leq u. \end{aligned}$$

Thus we have proved that

$$\tilde{u}_0 \in [0, u] = \{y \in H^1(\Omega) : 0 \leq y(z) \leq u(z) \ \text{for almost all } z \in \Omega\}, \quad \tilde{u}_0 \neq 0.$$

Therefore using (61), we see that equation (63) becomes

$$\begin{aligned}\langle A(\tilde{u}_0), h\rangle + \int_\Omega (\xi(z) + \chi)\tilde{u}_0 h\, dz + \int_{\partial\Omega} \beta(z)\tilde{u}_0 h\, d\sigma \\ = \int_\Omega \hat{c}\tilde{u}_0^{d-1} h\, dz - \int_\Omega c_9 \tilde{u}_0^{\eta-1} h\, dz, \quad \text{for all } h \in H^1(\Omega) \\ \Longrightarrow\ \tilde{u}_0 \text{ is a positive solution of (52),} \\ \Longrightarrow\ \tilde{u}_0 = \tilde{u} \in \operatorname{int} C_+ \quad \text{(see Proposition 3.12),} \\ \Longrightarrow\ \tilde{u} \leq u.\end{aligned}$$

COROLLARY 3.14. *Assume that hypotheses* $H(\xi)$, $H(\beta)$ *and* $H(f)$ *hold and* $D \subseteq \mathscr{L} = [\lambda_*, +\infty)$ *is bounded. Then there exists* $\tilde{u} \in \operatorname{int} C_+$ *such that* $\tilde{u} \leq u$, *for all* $u \in S(\lambda)$ *and all* $\lambda \in D$.

Now we are ready to produce the smallest positive solution of problem (P_λ).

PROPOSITION 3.15. *Assume that hypotheses* $H(\xi)$, $H(\beta)$ *and* $H(f)$ *hold and* $\lambda \in \mathscr{L} = [\lambda_*, +\infty)$. *Then problem* (P_λ) *admits a smallest positive solution* $u_\lambda^* \in \operatorname{int} C_+$.

PROOF. From Hu and Papageorgiou [5, Lemma 3.10, p. 178], we know that we can find $\{u_n\}_{n\geq 1} \subseteq S(\lambda)$ a decreasing sequence ($S(\lambda)$ is downward directed, see [3]) such that

$$\inf S(\lambda) = \inf_{n\geq 1} u_n.$$

We have

$$\langle A(u_n), h\rangle + \int_\Omega (\xi(z) + \lambda)u_n h\, dz + \int_{\partial\Omega} \beta(z)u_n h\, d\sigma = \int_\Omega f(z, u_n)h\, dz, \tag{64}$$
$$\text{for all } h \in H^1(\Omega),\ \text{all } n \in \mathbb{N},$$

$$\tilde{u} \leq u_n \leq u_1, \quad \text{for all } n \in \mathbb{N}. \tag{65}$$

Here $\tilde{u} \in \operatorname{int} C_+$ is the unique positive solution of (52) when $\chi = \lambda$ (see Proposition 3.13).

From (64) and (65) it follows that

$$\{u_n\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } H^1(\Omega) \quad \text{and} \quad u_n \to u_\lambda^* \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{66}$$

Hence, if in (64) we pass to the limit as $n \to \infty$ and use (66), then we obtain

$$\langle A(u_\lambda^*), h\rangle + \int_\Omega (\xi(z) + \lambda) u_\lambda^* h\, dz + \int_{\partial\Omega} \beta(z) u_\lambda^* h\, d\sigma = \int_\Omega f(z, u_\lambda^*) h\, dz, \quad \text{for all } h \in H^1(\Omega). \tag{67}$$

Also, from (65) and (66), we have

$$\tilde{u} \le u_\lambda^*. \tag{68}$$

From (67) and (68), we infer that

$$u_\lambda^* \in S(\lambda) \subseteq \operatorname{int} C_+ \quad \text{and} \quad u_\lambda^* = \inf S(\lambda).$$

Next we determine the monotonicity and continuity properties of the map $\lambda \mapsto u_\lambda^*$ from $\mathscr{L} = [\lambda_*, +\infty)$ into $C^1(\overline{\Omega})$.

PROPOSITION 3.16. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)'$ hold, then the map $\lambda \mapsto u_\lambda^*$ from $\mathscr{L} = [\lambda_*, +\infty)$ into $C^1(\overline{\Omega})$ is strictly decreasing (that is, if $\lambda < \eta$, then $u_\lambda^* - u_\eta^* \in \operatorname{int} C_+$) and right continuous.*

PROOF. From Proposition 3.7, we know that we can find $u_\eta \in S(\eta)$ such that

$$\begin{aligned} & u_\lambda^* - u_\eta \in \operatorname{int} C_+, \\ \Longrightarrow\ & u_\lambda^* - u_\eta^* \in \operatorname{int} C_+, \\ \Longrightarrow\ & \lambda \mapsto u_\lambda^* \text{ is strictly decreasing from } \mathscr{L} = [\lambda_*, +\infty) \text{ into } C^1(\Omega). \end{aligned}$$

Next let $\{\lambda_n\}_{n\ge1} \subseteq \mathscr{L}$ and suppose that $\lambda_n \to \lambda^+$. We have $\{u_{\lambda_n}^*\}_{n\ge1} \subseteq \operatorname{int} C_+$ is increasing and

$$\tilde{u} \le u_{\lambda_n}^* \le u_\lambda^*, \quad \text{for all } n \in \mathbb{N},$$

with $\tilde{u} \in \operatorname{int} C_+$ provided by Corollary 3.14 (with $D = \{\lambda_n, \lambda\}_{n\ge1}$). From the regularity theory (see Wang [17]), we can find $\alpha \in (0,1)$ and $c_{10} > 0$ such that

$$u_{\lambda_n}^* \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_{\lambda_n}^*\|_{C^{1,\alpha}(\overline{\Omega})} \le c_{10}, \quad \text{for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we may assume that

$$u_{\lambda_n}^* \to \hat{u} \quad \text{in } C^1(\overline{\Omega}). \tag{69}$$

Evidently $\hat{u} \in S(\lambda) \subseteq \operatorname{int} C_+$. We claim that $\hat{u} = u_\lambda^*$. Indeed, if this is not true, then using the strong maximum principle, we have

$$\begin{aligned} &\hat{u} - u_\lambda^* \in \operatorname{int} C_+ && \text{(see hypothesis } H(f)'(\text{v})\text{)}, \\ \Longrightarrow\ &u_{\lambda_n}^* - u_\lambda^* \in \operatorname{int} C_+, \quad \text{for all } n \geq n_0 && \text{(see (69))}, \end{aligned}$$

which contradicts Proposition 3.7. Therefore, $\hat{u} = u_\lambda^*$ and this proves the right continuity of the map $\lambda \mapsto u_\lambda^*$ from $\mathscr{L} = [\lambda_*, +\infty)$ into $C^1(\overline{\Omega})$.

So, summarizing, we can state the following bifurcation-type theorem describing the dependence of the set of positive solutions of problem (P_λ) on the parameter $\lambda > 0$.

THEOREM 3.17. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and $H(f)'$ hold. Then there exists $\lambda_* > 0$ such that*

(a) *for all $\lambda > \lambda_*$, problem (P_λ) has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+, \quad u_0 \neq \hat{u};$$

(b) *for $\lambda = \lambda_*$, problem (P_{λ_*}) has at least one positive solution $u_* \in \operatorname{int} C_+$;*

(c) *for all $\lambda \in (0, \lambda_*)$, problem (P_λ) has no positive solution;*

(d) *for every $\lambda \in \mathscr{L} = [\lambda_*, +\infty)$, problem (P_λ) has a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ and the map $\lambda \mapsto u_\lambda^*$ from $\mathscr{L} = [\lambda_*, +\infty)$ into $C^1(\overline{\Omega})$ is strictly increasing (that is, if $\lambda < \eta$, then $u_\lambda^* - u_\eta^* \in \operatorname{int} C_+$) and is right continuous.*

REMARK 3.18. An interesting problem is to extend the results of this work to equations driven by the p-Laplacian. A careful reading of the proof of Proposition 3.8, reveals that our approach here encounters difficulties when we try to extend it to nonlinear equations. So, new methods and techniques are necessary.

REFERENCES

1. Aizicovici, S., Papageorgiou, N. S., and Staicu, V., *Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints*, Mem. Amer. Math. Soc. 196 (2008), no. 915, 70 pp.
2. Ambrosetti, A., and Rabinowitz, P. H., *Dual variational methods in critical point theory and applications*, J. Functional Analysis 14 (1973), 349–381.
3. Filippakis, M. E., and Papageorgiou, N. S., *Multiple constant sign and nodal solutions for nonlinear elliptic equations with the p-Laplacian*, J. Differential Equations 245 (2008), no. 7, 1883–1922.
4. Gasiński, L., and Papageorgiou, N. S., *Nonlinear analysis*, Series in Mathematical Analysis and Applications, vol. 9, Chapman & Hall/CRC, Boca Raton, FL, 2006.

5. Hu, S., and Papageorgiou, N. S., *Handbook of multivalued analysis. Vol. I: Theory*, Mathematics and its Applications, vol. 419, Kluwer Academic Publishers, Dordrecht, 1997.
6. Kyritsi, S. T., and Papageorgiou, N. S., *Multiple solutions for superlinear Dirichlet problems with an indefinite potential*, Ann. Mat. Pura Appl. (4) 192 (2013), no. 2, 297–315.
7. Li, G., and Wang, C., *The existence of a nontrivial solution to a nonlinear elliptic problem of linking type without the Ambrosetti-Rabinowitz condition*, Ann. Acad. Sci. Fenn. Math. 36 (2011), no. 2, 461–480.
8. Motreanu, D., Motreanu, V. V., and Papageorgiou, N., *Topological and variational methods with applications to nonlinear boundary value problems*, Springer, New York, 2014.
9. Mugnai, D., *Addendum to: Multiplicity of critical points in presence of a linking: application to a superlinear boundary value problem, NoDEA. Nonlinear Differential Equations Appl. 11* (*2004*)*, no. 3, 379–391, and a comment on the generalized Ambrosetti-Rabinowitz condition*, NoDEA Nonlinear Differential Equations Appl. 19 (2012), no. 3, 299–301.
10. Papageorgiou, N. S., and Papalini, F., *Seven solutions with sign information for sublinear equations with unbounded and indefinite potential and no symmetries*, Israel J. Math. 201 (2014), no. 2, 761–796.
11. Papageorgiou, N. S., and Rădulescu, V. D., *Semilinear Neumann problems with indefinite and unbounded potential and crossing nonlinearity*, in "Recent trends in nonlinear partial differential equations. II. Stationary problems", Contemp. Math., vol. 595, Amer. Math. Soc., Providence, RI, 2013, pp. 293–315.
12. Papageorgiou, N. S., and Rădulescu, V. D., *Multiple solutions with precise sign for nonlinear parametric Robin problems*, J. Differential Equations 256 (2014), no. 7, 2449–2479.
13. Papageorgiou, N. S., and Rădulescu, V. D., *Multiplicity of solutions for resonant Neumann problems with an indefinite and unbounded potential*, Trans. Amer. Math. Soc. 367 (2015), no. 12, 8723–8756.
14. Papageorgiou, N. S., and Rădulescu, V. D., *Robin problems with indefinite and unbounded potential, resonant at* $-\infty$*, superlinear at* $+\infty$, Tohoku Math. J. (2) 69 (2017), no. 2, 261–286.
15. Papageorgiou, N. S., and Smyrlis, G., *On a class of parametric Neumann problems with indefinite and unbounded potential*, Forum Math. 27 (2015), no. 3, 1743–1772.
16. Qin, D., Tang, X., and Zhang, J., *Multiple solutions for semilinear elliptic equations with sign-changing potential and nonlinearity*, Electron. J. Differential Equations 2013 (2013), no. 207, 1–9.
17. Wang, X.-J., *Neumann problems of semilinear elliptic equations involving critical Sobolev exponents*, J. Differential Equations 93 (1991), no. 2, 283–310.
18. Zhang, Q., and Liu, C., *Multiple solutions for a class of semilinear elliptic equations with general potentials*, Nonlinear Anal. 75 (2012), no. 14, 5473–5481.

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