MATH. SCAND. 122 (2018), 141–150

COMPOSITION OPERATORS ON WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS ON THE UPPER HALF PLANE

WOLFGANG LUSKY

Abstract

We consider moderately growing weight functions v on the upper half plane $\mathbb{G}$ called normal weights which include the examples $(\operatorname{Im} w)^a$, $w \in \mathbb{G}$, for fixed $a > 0$. In contrast to the comparable, well-studied situation of normal weights on the unit disc here there are always unbounded composition operators C_φ on the weighted spaces $Hv(\mathbb{G})$. We characterize those holomorphic functions $\varphi\colon \mathbb{G} \to \mathbb{G}$ where the composition operator C_φ is a bounded operator $Hv(\mathbb{G}) \to Hv(\mathbb{G})$ by a simple property which depends only on φ but not on v. Moreover we show that there are no compact composition operators C_φ on $Hv(\mathbb{G})$.

1. Introduction

Let $O \subset \mathbb{C}$ be open, non-empty and consider a continuous function $v\colon O \to]0, \infty[$. Put

$$Hv(O) = \Big\{ h\colon O \to \mathbb{C} : h \text{ holomorphic}, \ \ \|h\|_v := \sup_{w \in O} |h(w)|v(w) < \infty \Big\}.$$

In other words, the growth of a (not necessarily bounded) function $h \in Hv(O)$ is controlled by $1/v$.

For a holomorphic function $\varphi\colon O \to O$ we define the composition operator C_φ on $Hv(O)$ by $C_\varphi h = h \circ \varphi$, $h \in Hv(O)$. Classical examples of O are the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and a half space, e.g. $\mathbb{G} := \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$.

It is of some interest to find similarities and differences between the weighted spaces over $\mathbb{D}$ and over $\mathbb{G}$ as far as Banach space properties are concerned.

There is an extensive number of papers dealing with 'typical' weights v on $\mathbb{D}$ where v satisfies $v(z) = v(|z|)$, $z \in \mathbb{D}$, $v(t) \le v(s)$ if $0 \le s \le t < 1$, and $\lim_{r \to 1} v(r) = 0$ (e.g. see [3], [4], [7], [8], [9], [10], [11]). A typical weight

Received 21 December 2015.
DOI: https://doi.org/10.7146/math.scand.a-97126

v is *normal* if it satisfies

$$\sup_{k\in\mathbb{N}} \frac{v(1-2^{-k})}{v(1-2^{-k-1})} < \infty \tag{1.1}$$

and

$$\inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{N}} \frac{v(1-2^{-k-n})}{v(1-2^{-k})} < 1. \tag{1.2}$$

Standard examples are the weights $v(z) = (1-|z|)^a$ for some $a > 0$. If v is normal then the Banach space $Hv(\mathbb{D})$ is isomorphic to ℓ_∞, the space of all bounded sequences [7], [10]. If v is typical and satisfies (1.1) but not (1.2) then $Hv(\mathbb{D})$ is isomorphic to H_∞, the space of all bounded holomorphic functions on $\mathbb{D}$ [8].

There are similar results for weighted spaces over $\mathbb{G}$.

DEFINITION 1.1. Let $v\colon \mathbb{G} \to]0,\infty[$ be continuous.

(i) v is called a *standard weight* if $v(w) = v(i \operatorname{Im} w)$, $w \in \mathbb{G}$, $v(is) \le v(it)$ whenever $0 < s \le t < \infty$, and $\lim_{s\to 0} v(is) = 0$.

(ii) A standard weight v on $\mathbb{G}$ is called *normal* if it satisfies

$$\sup_{k\in\mathbb{Z}} \frac{v(2^{k+1}i)}{v(2^k i)} < \infty \tag{1.3}$$

and

$$\inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{v(2^k i)}{v(2^{k+n} i)} < 1. \tag{1.4}$$

For example the weights $(\operatorname{Im} w)^a$, for some $a > 0$, are normal weights on $\mathbb{G}$.

Again, $Hv(\mathbb{G})$ is isomorphic to ℓ_∞ if v is normal [2]. If v is a standard weight on $\mathbb{G}$ satisfying (1.3) but not (1.4) then $Hv(\mathbb{G})$ is isomorphic (as a Banach space) to H_∞ [6]. However, the situation over $\mathbb{G}$ cannot be reduced to the one over $\mathbb{D}$ by simply considering $v \circ \psi$ for a conformal map $\psi\colon \mathbb{D} \to \mathbb{G}$. Indeed, $v \circ \psi$ is not typical over $\mathbb{D}$ even if v is standard over $\mathbb{G}$.

The similarities between weighted spaces over $\mathbb{D}$ and $\mathbb{G}$ completely break down if we consider composition operators. It was shown in [4, Theorem 2.3] (together with the fact that normal weights are essential – see [3] and Section 2 below) that, for normal weights v over $\mathbb{D}$, the composition operator C_φ is a bounded operator $Hv(\mathbb{D}) \to Hv(\mathbb{D})$ for any holomorphic function $\varphi\colon \mathbb{D} \to \mathbb{D}$. Moreover there are always compact composition operators $Hv(\mathbb{D}) \to Hv(\mathbb{D})$.

The purpose of this paper is to show that the situation over $\mathbb{G}$ is entirely different. There are always unbounded composition operators even if v is normal over $\mathbb{G}$. For normal weights we give a complete characterisation of the holomorphic functions $\varphi\colon \mathbb{G} \to \mathbb{G}$ such that C_φ is bounded. It also shows that the boundedness of C_φ does not depend on special properties of the given weight. Moreover we prove that there are no compact composition operators on $Hv(\mathbb{G})$.

The main result of the paper is the following.

THEOREM 1.2. *Let v be a normal weight on $\mathbb{G}$ and $\varphi\colon \mathbb{G} \to \mathbb{G}$ a holomorphic function. Then C_φ is a bounded operator $Hv(\mathbb{G}) \to Hv(\mathbb{G})$ if and only if*

$$\sup_{w\in\mathbb{G}} \frac{\operatorname{Im} w}{\operatorname{Im} \varphi(w)} < \infty. \tag{1.5}$$

We prove Theorem 1.2 in Section 2. Here we discuss some

EXAMPLES 1.3. Let v be a normal weight on $\mathbb{G}$. Then, according to (1.4), v is unbounded. Hence, C_φ cannot be bounded if φ is constant on $\mathbb{G}$. Let $\varphi_1(w) = -1/w$, $\varphi_2(w) = w - 1/w$, $\varphi_3(w) = \log(w)$ (main branch), $w \in \mathbb{G}$. Then all φ_k are holomorphic and satisfy $\varphi_k(\mathbb{G}) \subset \mathbb{G}$. In view of (1.5), C_{φ_2} is bounded while C_{φ_k} are unbounded if $k = 1, 3$.

As a consequence of Theorem 1.2 we obtain

THEOREM 1.4. *Let v be a normal weight on $\mathbb{G}$. Then there is no holomorphic map $\varphi\colon \mathbb{G} \to \mathbb{G}$ such that the composition operator $C_\varphi\colon Hv(\mathbb{G}) \to Hv(\mathbb{G})$ is compact.*

We prove Theorem 1.4 in Section 3. Here we discuss another consequence of Theorem 1.2. To this end put

$$\mathscr{C}(v) = \{\varphi\colon \mathbb{G} \to \mathbb{G} \text{ holomorphic} : C_\varphi : Hv(\mathbb{G}) \to Hv(\mathbb{G}) \text{ bounded}\}.$$

In fact, $\mathscr{C}(v)$ is a cone and has a certain ideal property with respect to addition.

COROLLARY 1.5. *Let v be a normal weight on $\mathbb{G}$. Then:*

(a) *For $\alpha, \beta > 0$ and $\varphi, \psi \in \mathscr{C}(v)$ we have $\alpha\varphi + \beta\psi \in \mathscr{C}(v)$.*

(b) *If $\varphi\colon \mathbb{G} \to \mathbb{G}$ is holomorphic and $\psi \in \mathscr{C}(v)$ then $\varphi + \psi \in \mathscr{C}(v)$.*

Corollary 1.5 is a direct consequence of (1.5). So we obtain that, for every $\epsilon > 0$ and every holomorphic function $\varphi\colon \mathbb{G} \to \mathbb{G}$, with $\psi = \epsilon \,\mathrm{id}_{\mathbb{G}} + \varphi$, the composition operator C_ψ is bounded on $Hv(\mathbb{G})$. In particular, $\mathscr{C}(v)$ is dense in $\{\varphi\colon \mathbb{G} \to \mathbb{G} : \varphi \text{ holomorphic}\}$ with respect to the topology of compact convergence. (1.5) also shows that $\mathscr{C}(v)$ does not depend on special properties of v. In fact, for all normal weights the set $\mathscr{C}(v)$ is the same.

Finally, we pose the following

OPEN QUESTION. Let v be a standard weight on $\mathbb{G}$. Assume that

$$\mathscr{C}(v) = \left\{\varphi\colon \mathbb{G} \to \mathbb{G} \text{ holomorphic} : \sup_{w\in\mathbb{G}} \frac{\operatorname{Im} w}{\operatorname{Im}\varphi(w)} < \infty\right\}.$$

Does it follow that v is normal?

In Section 4 we discuss a result which might suggest that there is a positive answer.

2. Proof of Theorem 1.2

We start with a well-known lemma [2, Lemma 1.6].

LEMMA 2.1. *Let v be a standard weight on $\mathbb{G}$. Then* (1.3) *holds if and only if there are constants $c > 0$ and $a > 0$ such that*

$$\frac{v(it)}{v(is)} \le c\left(\frac{t}{s}\right)^a \tag{2.1}$$

whenever $0 < s \le t$.

Condition (1.4) *holds if and only if there are constants $d > 0$ and $b > 0$ such that*

$$d\left(\frac{t}{s}\right)^b \le \frac{v(it)}{v(is)} \tag{2.2}$$

whenever $0 < s \le t$.

We immediately obtain

PROPOSITION 2.2. *Let v be a standard weight on $\mathbb{G}$ satisfying (1.3) and $\varphi\colon \mathbb{G} \to \mathbb{G}$ a holomorphic map satisfying (1.5). Then C_φ is bounded on $Hv(\mathbb{G})$.*

PROOF. Let $h \in Hv(\mathbb{G})$. For any $w \in \mathbb{G}$ we have, with the constants a and c of (2.1),

$$\begin{aligned}
|(C_\varphi h)(w)|v(w) &= |h(\varphi(w))|v(w)\\
&= |h(\varphi(w))|v(\varphi(w))\frac{v(w)}{v(\varphi(w))}\\
&\le \|h\|_v \cdot \begin{cases} c\left(\dfrac{\operatorname{Im} w}{\operatorname{Im}\varphi(w)}\right)^a & \text{if } \operatorname{Im} w \ge \operatorname{Im}\varphi(w)\\ 1 & \text{otherwise}\end{cases}\\
&\le \|h\|_v d
\end{aligned}$$

where d is a constant which does not depend on h or w. Here we used (2.1), (1.5) and the fact that $v(it)$ is increasing in t. This shows that C_φ is bounded.

To show the converse of Proposition 2.2 we need the notion of associated weight. Let v be a weight on $\mathbb{G}$. Then the associated weight $\tilde{v}$ is defined by

$$\tilde{v}(w) = \inf\left\{ \frac{1}{|h(w)|} : h \in Hv(\mathbb{G}),\ \|h\|_v \le 1 \right\}.$$

(The definition of associated weight can be extended to weights on arbitrary open subsets of $\mathbb{C}$.) We have $v(w) \le \tilde{v}(w)$ for all $w \in \mathbb{G}$. If also $\tilde{v} \le dv$ for some constant d then v is called essential weight.

LEMMA 2.3. *Let v be a standard weight on $\mathbb{G}$ satisfying* (1.3). *Then there is a constant $c > 0$ such that, for every $w \in \mathbb{G}$, there exists $h \in Hv(\mathbb{G})$ with $\|h\|_v = 1$ and $|h(w)|v(w) \ge c$.*

PROOF. It is well-known that, for every $w \in \mathbb{G}$ there is $h \in Hv(\mathbb{G})$ with $|h(w)|\tilde{v}(w) = \|h\|_v = 1$ [3]. Moreover, if v is a standard weight with (1.3) then v is essential [1, Theorem 1.3 and Proposition 3.5] which immediately proves the lemma.

PROPOSITION 2.4. *Let v be a normal weight on $\mathbb{G}$ and assume that $\varphi\colon \mathbb{G} \to \mathbb{G}$ is a holomorphic map such that C_φ is a bounded operator on $Hv(\mathbb{G})$. Then φ satisfies* (1.5).

PROOF. Fix $w \in \mathbb{G}$ and find $h \in Hv(\mathbb{G})$ with $\|h\|_v = 1$ and $|h(\varphi(w)|v(\varphi(w)) \ge c$ where c is the universal constant of Lemma 2.3. If $\operatorname{Im} w \le \operatorname{Im} \varphi(w)$ then $\operatorname{Im} w/\operatorname{Im} \varphi(w) \le 1$. Now assume $\operatorname{Im} w \ge \operatorname{Im} \varphi(w)$. Then we obtain with the constants of (2.2)

$$\begin{aligned}
\|C_\varphi\| &\ge \|C_\varphi(h)\|_v \\
&\ge |h(\varphi(w))|v(w) \\
&= |h(\varphi(w))|v(\varphi(w))\frac{v(w)}{v(\varphi(w))} \\
&\ge cd\left(\frac{\operatorname{Im} w}{\operatorname{Im} \varphi(w)}\right)^b
\end{aligned}$$

which implies that $\operatorname{Im} w/\operatorname{Im} \varphi(w) \le (\|C_\varphi\|/cd)^{1/b}$. This shows that φ satisfies (1.5).

The proof of Theorem 1.2 follows from Propositions 2.2 and 2.4.

3. Compact composition operators

Here we use that $\mathbb{G}$ and $\mathbb{D}$ are conformally equivalent. Consider

$$\alpha(z) = \frac{1+z}{1-z} i \quad \text{for } z \neq 1 \qquad \text{and} \qquad \beta(w) = \frac{w-i}{w+i} \quad \text{for } w \neq -i.$$

Then α and β are holomorphic and we obtain

$$\alpha \circ \beta|_{\mathbb{G}} = \mathrm{id}_{\mathbb{G}} \qquad \text{and} \qquad \beta \circ \alpha|_{\mathbb{D}} = \mathrm{id}_{\mathbb{D}}\,. \tag{3.1}$$

First we show that the growth along the lines parallel to the imaginary axis of the imaginary part of a holomorphic function mapping $\mathbb{G}$ into $\mathbb{G}$ is at most linear.

LEMMA 3.1. *Let $\varphi: \mathbb{G} \to \mathbb{G}$ be holomorphic. Then there is a constant $c(\varphi) > 0$ such that*

$$\frac{t}{\operatorname{Im} \varphi(x+it)} \geq c(\varphi) \qquad \textit{whenever} \quad x \in \mathbb{R} \quad \textit{and} \quad t \geq \sqrt{x^2+1}. \tag{3.2}$$

PROOF. (a) First we assume in addition that $\varphi(i) = i$. Put $\psi = \beta \circ \varphi \circ \alpha|_{\mathbb{D}}$. Then ψ is holomorphic and satisfies $\psi(\mathbb{D}) \subset \mathbb{D}$ and $\psi(0) = 0$. The Schwarz lemma yields $|\psi(z)| \leq |z|$ for all $z \in \mathbb{D}$. With (3.1) this implies

$$\left|\frac{\varphi(w)-i}{\varphi(w)+i}\right|^2 \leq \left|\frac{w-i}{w+i}\right|^2 \qquad \text{for all} \quad w \in \mathbb{G}$$

from which we obtain

$$\begin{aligned}(|\varphi(w)|^2 + 1 - 2\operatorname{Im}\varphi(w))(|w|^2 + 1 + 2\operatorname{Im} w)& \\ \leq (|\varphi(w)|^2 + 1 + 2\operatorname{Im}\varphi(w))(|w|^2 + 1 - 2\operatorname{Im} w)&.\end{aligned}$$

We conclude

$$\frac{\operatorname{Im} w}{\operatorname{Im}\varphi(w)} \leq \frac{|w|^2+1}{|\varphi(w)|^2+1} \leq \frac{|w|^2+1}{(\operatorname{Im}\varphi(w))^2}, \qquad w \in \mathbb{G}. \tag{3.3}$$

Now fix $x \in \mathbb{R}$. Then we have $2t^2 \geq t^2 + x^2 + 1$ for all $t \geq \sqrt{x^2+1}$. (3.3) yields

$$\frac{t}{\operatorname{Im}\varphi(x+it)} \leq \frac{t^2+x^2+1}{(\operatorname{Im}\varphi(x+it))^2} \leq 2\frac{t^2}{(\operatorname{Im}\varphi(x+it))^2}$$

and hence
$$\frac{1}{2} \leq \frac{t}{\operatorname{Im}\varphi(x+it)} \qquad \text{for all} \quad t \geq \sqrt{x^2+1}.$$

(b) Now let φ be arbitrary. Then put

$$\varphi_1(w) = \frac{\varphi(w)}{\operatorname{Im}\varphi(i)} - \frac{\operatorname{Re}\varphi(i)}{\operatorname{Im}\varphi(i)}, \qquad w \in \mathbb{G}.$$

(Take into account that $\operatorname{Im}\varphi(i) > 0$ since $\varphi(i) \in \mathbb{G}$.) φ_1 is holomorphic and we have $\varphi_1(\mathbb{G}) \subset \mathbb{G}$ and $\varphi_1(i) = i$. Hence (a) implies that, for every $x \in \mathbb{R}$,

$$\frac{t}{\operatorname{Im}\varphi_1(x+it)} \geq \frac{1}{2} \qquad \text{whenever} \quad t \geq \sqrt{x^2+1}.$$

Then φ satisfies (3.2) with $c(\varphi) = 1/(2\operatorname{Im}\varphi(i))$ since $\operatorname{Im}\varphi(w) = \operatorname{Im}\varphi_1(w) \cdot \operatorname{Im}\varphi(i)$.

LEMMA 3.2. *Let $w_n \in \mathbb{G}$ be such that* $\lim_{n\to\infty} |w_n| = \infty$. *Then there are a subsequence (w_{m_n}) and holomorphic functions $f_n\colon \mathbb{G} \to \mathbb{C}$ with*

$$\sup_n \sup_{w\in\mathbb{G}} |f_n(w)| < \infty \qquad \textit{and} \qquad f_n(w_{m_k}) = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

PROOF. We use again the map β from (3.1). Consider $z_n = \beta(w_n)$. By our assumption on (w_n) we have $\lim_{n\to\infty} |z_n| = 1$. Pick a subsequence (z_{m_n}) such that $1 - |z_{m_{n+1}}| \leq (1 - |z_{m_n}|)/2$ for each n. Then (z_{m_n}) is an interpolating sequence [5, Theorem 9.1 and Theorem 9.2]. This means that, for every bounded function $\tilde{g}$ on $\Omega = \{z_{m_n} : n = 1, 2, \ldots\}$, there is a holomorphic function g on $\mathbb{D}$ with $g|_\Omega = \tilde{g}$ and $\sup_{\mathbb{D}} |g(z)| \leq c \sup_\Omega |\tilde{g}|$ where $c > 0$ is a universal constant. In particular there are holomorphic functions $g_n\colon \mathbb{D} \to \mathbb{C}$ with

$$\sup_n \sup_{\mathbb{D}} |g_n(z)| < \infty \qquad \text{and} \qquad g_n(z_{m_k}) = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

Finally, take $f_n(w) = g_n(\beta(w))$, $w \in \mathbb{G}$.

The following lemma is obvious.

LEMMA 3.3. *Let v be a weight on $\mathbb{G}$ and let $h_n \in Hv(\mathbb{G})$. Assume that there are $w_n \in \mathbb{G}$ and a constant $c > 0$ with $|h_n(w_n) - h_m(w_n)|v(w_n) \geq c$ for all n and $m \neq n$. Then (h_n) does not have a norm convergent subsequence.*

PROPOSITION 3.4. *Let v be a normal weight on $\mathbb{G}$ and let $\varphi\colon \mathbb{G} \to \mathbb{G}$ be a holomorphic function satisfying* (1.5). *Then there is a sequence of holomorphic functions $h_n \in Hv(\mathbb{G})$ with $\sup_n \|h_n\|_v < \infty$ such that $(C_\varphi h_n)$ does not contain any convergent subsequence.*

PROOF. Fix $t_n > 0$ with $\lim_{n\to\infty} t_n = \infty$. Put $w_n = \varphi(it_n)$. In view of (1.5) we have $\sup_n(t_n/\operatorname{Im}\varphi(it_n)) < \infty$. Hence $\infty = \lim_{n\to\infty}\operatorname{Im}\varphi(it_n) = \lim_{n\to\infty}|w_n|$.

In view of Lemma 3.2, by perhaps going over to a subsequence, we can assume that there are holomorphic functions $f_n\colon \mathbb{G}\to\mathbb{C}$ with

$$\sup_n \sup_{\mathbb{G}} |f_n(w)| < \infty \qquad \text{and} \qquad f_n(w_k) = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

Since v is normal, according to Lemma 2.3, we find $\tilde{h}_n \in Hv(\mathbb{G})$ with $c_1 \le |\tilde{h}_n(w_n)|v(w_n) \le \|\tilde{h}_n\|_v = 1$ for all n. Here $c_1 > 0$ is a constant. Put $h_n(w) = f_n(w)\tilde{h}_n(w)$, $w \in \mathbb{G}$. Then $\sup_n \|h_n\|_v < \infty$. We obtain, for $n \neq m$,

$$\begin{aligned}|(C_\varphi h_n)(it_n) - (C_\varphi h_m)(it_n)|v(it_n) &= |h_n(w_n) - h_m(w_n)|v(it_n)\\ &= |h_n(w_n)|v(w_n)\frac{v(it_n)}{v(\varphi(it_n))}\\ &\ge c_1\frac{v(it_n)}{v(\varphi(it_n))}.\end{aligned}$$

Let a, b, c, d be the constants of (2.1) and (2.2) and consider the constant $c(\varphi)$ of Lemma 3.1. If $t_n \ge \operatorname{Im}\varphi(it_n)$ then (2.2) implies

$$\begin{aligned}\frac{v(it_n)}{v(\varphi(it_n))} &\ge d\left(\frac{t_n}{\operatorname{Im}\varphi(it_n)}\right)^b\\ &\ge dc(\varphi)^b\end{aligned}$$

for all n such that $t_n \ge 1$. (We applied Lemma 3.1 for $x = 0$.) If $t_n \le \operatorname{Im}\varphi(it_n)$ then (2.1) implies

$$\begin{aligned}\frac{v(it_n)}{v(\varphi(it_n))} &\ge \frac{1}{c}\left(\frac{t_n}{\operatorname{Im}\varphi(it_n)}\right)^a\\ &\ge \frac{1}{c}c(\varphi)^a.\end{aligned}$$

for all n such that $t_n \ge 1$. Thus

$$\big|(C_\varphi h_n)(it_n) - (C_\varphi h_m)(it_n)\big|v(it_n) \ge c_1 \min\left(\frac{c(\varphi)^a}{c}, dc(\varphi)^b\right)$$

for all $m \neq n$ and n such that $t_n \ge 1$. In view of Lemma 3.3 the sequence $(C_\varphi h_n)$ cannot have a convergent subsequence.

Proposition 3.4 shows that no composition operator on $Hv(\mathbb{G})$ for normal v can be compact.

4. Concluding remarks

Let v be a standard weight on $\mathbb{G}$. Then it can happen that $Hv(\mathbb{G}) = \{0\}$. However $Hv(\mathbb{G}) \neq \{0\}$ if and only if there are constants $a > 0$ and $b > 0$ with $v(it) \leq ae^{bt}$ for all $t > 0$ [12]. Hence, in view of Lemma 2.1, for standard weights satisfying (1.3) we always have $Hv(\mathbb{G}) \neq \{0\}$.

Now let v be an arbitrary standard weight on $\mathbb{G}$ with $Hv(\mathbb{G}) \neq \{0\}$ and consider the associated weight $\tilde{v}$. Then $\tilde{v}$ is a standard weight, too. Indeed, we have $\tilde{v}(w) = \tilde{v}(i \operatorname{Im} w)$ by definition. Moreover, $\tilde{v}(it) \geq \tilde{v}(is)$ whenever $0 < s \leq t$ according to [1], Lemma 2.1. Finally, for $f_n(w) = e^{inw}$, $w \in \mathbb{G}$, there are $t_n \to 0$ with $e^{-nt_n} v(it_n) = \sup_{t>0} |f_n(it)|v(it) = \|f_n\|_v$ ([1], Lemma 3.1). This implies $v(it_n) = \tilde{v}(it_n)$ for all n. We have $\lim_{n\to\infty} \tilde{v}(it_n) = 0$ and hence, together with the preceding property, $\lim_{t\to 0} \tilde{v}(it) = 0$.

We get

THEOREM 4.1. *Let v be a standard weight with $Hv(\mathbb{G}) \neq \{0\}$. Then*

$$\left\{\varphi\colon \mathbb{G} \to \mathbb{G} \text{ holomorphic} : \sup_{w\in\mathbb{G}} \frac{\operatorname{Im} w}{\operatorname{Im} \varphi(w)} < \infty\right\} \subset \mathscr{C}(v)$$

if and only if $\tilde{v}$ satisfies (1.3).

PROOF. At first assume

$$\left\{\varphi\colon \mathbb{G} \to \mathbb{G} \text{ holomorphic} : \sup_{w\in\mathbb{G}} \frac{\operatorname{Im} w}{\operatorname{Im} \varphi(w)} < \infty\right\} \subset \mathscr{C}(v).$$

Let $\varphi(w) = w/2$, $w \in \mathbb{G}$. Then φ is holomorphic and $\varphi(\mathbb{G}) \subset \mathbb{G}$. By assumption, C_φ is bounded on $Hv(\mathbb{G})$. So for each $t > 0$ there is a function $h \in Hv(\mathbb{G})$ with $1 = \|h\|_v = |h(\varphi(it))|\tilde{v}(\varphi(t))$ and

$$\begin{aligned}\frac{\tilde{v}(it)}{\tilde{v}(it/2)} &= |h(\varphi(it))|\tilde{v}(\varphi(t))\frac{\tilde{v}(it)}{\tilde{v}(it/2)} \\ &= |h(\varphi(it))|\tilde{v}(it) \\ &\leq \|C_\varphi\|.\end{aligned}$$

In particular, $\tilde{v}(i2^{k+1})/\tilde{v}(i2^k) \leq \|C_\varphi\|$ for all $k \in \mathbb{Z}$.

Conversely, let $\tilde{v}$ satisfy (1.3) and let $\varphi\colon \mathbb{G} \to \mathbb{G}$ be holomorphic and satisfy (1.5). Then Proposition 2.2, applied to $\tilde{v}$ instead of v, shows that C_φ is bounded on $H\tilde{v}(\mathbb{G}) = Hv(\mathbb{G})$. Hence

$$\left\{\varphi\colon \mathbb{G} \to \mathbb{G} \text{ holomorphic} : \sup_{w\in\mathbb{G}} \frac{\operatorname{Im} w}{\operatorname{Im} \varphi(w)} < \infty\right\} \subset \mathscr{C}(v).$$

Theorem 4.1 for standard weights on $\mathbb{G}$ seems to be the equivalent of [4, Theorem 2.3] for typical weights on $\mathbb{D}$.

REFERENCES

1. Ardalani, M. A., and Lusky, W., *Bounded operators on weighted spaces of holomorphic functions on the upper half-plane*, Studia Math. 209 (2012), no. 3, 225–234.
2. Ardalani, M. A., and Lusky, W., *Weighted spaces of holomorphic functions on the upper halfplane*, Math. Scand. 111 (2012), no. 2, 244–260.
3. Bierstedt, K. D., Bonet, J., and Taskinen, J., *Associated weights and spaces of holomorphic functions*, Studia Math. 127 (1998), no. 2, 137–168.
4. Bonet, J., Domański, P., Lindström, M., and Taskinen, J., *Composition operators between weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. Ser. A 64 (1998), no. 1, 101–118.
5. Duren, P. L., *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970.
6. Harutyunyan, A., and Lusky, W., *A remark on the isomorphic classification of weighted spaces of holomorphic functions on the upper half plane*, Ann. Univ. Sci. Budapest. Sect. Comput. 39 (2013), 125–135.
7. Lusky, W., *On weighted spaces of harmonic and holomorphic functions*, J. London Math. Soc. (2) 51 (1995), no. 2, 309–320.
8. Lusky, W., *On the isomorphic classification of weighted spaces of holomorphic functions*, Acta Univ. Carolin. Math. Phys. 41 (2000), no. 2, 51–60.
9. Shields, A. L., and Williams, D. L., *Bonded projections, duality, and multipliers in spaces of analytic functions*, Trans. Amer. Math. Soc. 162 (1971), 287–302.
10. Shields, A. L., and Williams, D. L., *Bounded projections, duality, and multipliers in spaces of harmonic functions*, J. Reine Angew. Math. 299/300 (1978), 256–279.
11. Shields, A. L., and Williams, D. L., *Bounded projections and the growth of harmonic conjugates in the unit disc*, Michigan Math. J. 29 (1982), no. 1, 3–25.
12. Stanev, M. A., *Weighted Banach spaces of holomorphic functions in the upper half plane*, preprint arxiv:math/9911082 [math.FA], 1999.

INSTITUTE OF MATHEMATICS
PADERBORN UNIVERSITY
WARBURGER STRAßE 100
D-33098 PADERBORN
GERMANY
E-mail: lusky@math.upb.de