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A GLOBAL BRIANÇON-SKODA-HUNEKE-SZNAJDMAN THEOREM

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Abstract

We prove a global effective membership result for polynomials on a non-reduced algebraic subvariety of $\mathbb{C}^N$. It can be seen as a global version of a recent local result of Sznajdman, generalizing the Briançon-Skoda-Huneke theorem for the local ring of holomorphic functions at a point on a reduced analytic space.

1. Introduction

Let x be a point on a smooth analytic variety X of pure dimension n and let $\mathscr{O}_x$ be the local ring of holomorphic functions. The classical Briançon-Skoda theorem [26] states that if $(a) = (a_1, \ldots, a_m)$ is any ideal in $\mathscr{O}_x$ and ϕ is in $\mathscr{O}_x$, then $\phi \in (a)^r$ if

$$|\phi| \le C|a|^{\nu+r-1} \tag{1.1}$$

holds with $\nu = \min(m, n)$. The proof given in [26] is purely analytic. However, condition (1.1) is equivalent to saying that ϕ belongs to the the integral closure $\overline{(a)^{\nu+r-1}}$, and thus the theorem admits a purely algebraic formulation. Therefore it was somewhat astonishing that it took several years before algebraic proofs were found [21], [22]. Later on, Huneke [18] proved a far-reaching algebraic generalization which contains the following statement for non-smooth X.

> *Let $x \in X$ be a point on a reduced analytic variety of pure dimension. There is a number ν such that if $(a) = (a_1, \ldots, a_m)$ is any ideal in $\mathscr{O}_x$ and ϕ is in $\mathscr{O}_x$, then* (1.1) *implies that $\phi \in (a)^r$.*

An important point is that ν is uniform with respect to both (a) and r. The smallest possible such ν is called the Briançon-Skoda number, and it depends on the complexity of the singularities of X at x. An analytic proof of this statement appeared in [4]. A nice variant for a non-reduced X of pure dimension is formulated and proved in [27].

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Let x be a point on a non-reduced analytic space X of pure dimension n, and let X_{red} be the underlying reduced space, cf. Section 2.1 below. There is a natural surjective mapping $\mathscr{O}_{X,x} \to \mathscr{O}_{X_{\mathrm{red}},x}$. Let $i\colon X \to \Omega \subset \mathbb{C}^N$ be a local embedding, and let $\mathscr{J}_{X,x}$ be the associated local ideal in $\mathscr{O}_{\Omega,x}$, so that $\mathscr{O}_x = \mathscr{O}_{X,x} = \mathscr{O}_{\Omega,x}/\mathscr{J}_{X,x}$. A holomorphic differential operator L in Ω is Noetherian at x if $L\phi$ vanishes on $X_{\mathrm{red},x}$ (or equivalently, $L\phi \in \sqrt{\mathscr{J}_{X,x}} = \mathscr{J}_{X_{\mathrm{red}},x}$) for all $\phi \in \mathscr{J}_{X,x}$. Such an L defines an intrinsic mapping

$$L\colon \mathscr{O}_{X,x} \to \mathscr{O}_{X_{\mathrm{red}},x}, \quad \phi \mapsto L\phi.$$

THEOREM 1.1 (Sznajdman, [27]). *Given $x \in X$, there is a finite set L_α of Noetherian operators at x and a number ν such that for each ideal $(a) = (a_1, \ldots, a_m) \subset \mathscr{O}_{X,x}$ and $\phi \in \mathscr{O}_{X,x}$,*

$$|L_\alpha \phi| \le C|a|^{\nu+r} \quad \text{on } X_{\mathrm{red},x} \tag{1.2}$$

for all α, implies that $\phi \in (a)^r$.

Here $|a|$ means $|a_1| + \cdots + |a_m|$ (where $|a_j|$ is the modulus of the image of a_j in $\mathscr{O}_{X,x}$), which up to constants is independent of the choice of generators of the ideal (a). The condition (1.2) means that $L_\alpha\phi$ is in the integral closure of the image in $\mathscr{O}_{X_{\mathrm{red}},x}$ of $(a)^{\nu+r}$.

Applying to $(a) = (0)$ we find that $L_\alpha\phi = 0$ on $X_{\mathrm{red},x}$ for all α implies that $\phi = 0$ in $\mathscr{O}_{X,x}$.

We now turn our attention to global variants. Let V be a purely n-dimensional algebraic subvariety of $\mathbb{C}^N$ and let $J_V \subset \mathbb{C}[x_1, \ldots, x_N]$ be the associated ideal. Assume that F_j are polynomials in $\mathbb{C}^N$ of degree $\le d$. If the polynomial Φ belongs to the restriction of the ideal $(F_1, \ldots, F_m)$ to V, i.e., there are polynomials Q_j such that

$$\Phi = \sum_1^m F_j Q_j + J_V, \tag{1.3}$$

then it is natural to ask for a representation (1.3) with some control of the degree of Q_j. It is well-known that if $V = \mathbb{C}^N$, then in general $\max_j \deg F_j Q_j$ must be doubly exponential in d, i.e., like 2^{2^d}. However, in the Nullstellensatz, i.e., $\Phi = 1$, then (roughly speaking) d^n is enough, this is due to Kollár [20] and Jelonek, [19]. In [17] Hickel proved a global effective version of the Briançon-Skoda theorem for polynomial ideals in $\mathbb{C}^n$, basically saying that if $|\Phi|/|F|^{\min(m,n)}$ is locally bounded, then there is a representation (1.3) in $\mathbb{C}^n$ with $\deg F_j Q_j \le \deg \Phi + Cd^n$. For the precise statement, see [17] or [8]. In [8, Theorem A] a generalization to polynomials on reduced algebraic subvarieties of $\mathbb{C}^N$ appeared. Our objective in this paper is to find a generalization to a not necessarily reduced algebraic subvariety V of $\mathbb{C}^N$ of pure dimension n.

Let X be the closure (see Section 2.2) of V in $\mathbb{P}^N$ and let X_{red} be the underlying reduced variety. Given polynomials $F_1, \dots, F_m$, let f_j denote the corresponding d-homogenizations, considered as sections of the line bundles $\mathcal{O}(d)|_{X_{\text{red}}}$, and let $\mathcal{J}_f$ be the coherent analytic sheaf on X_{red} generated by f_j. Furthermore, let c_∞ be the maximal codimension of the so-called *distinguished varieties* of the sheaf $\mathcal{J}_f$, in the sense of Fulton-MacPherson, that are contained in

$$X_{\text{red},\infty} := X_{\text{red}} \setminus V_{\text{red}},$$

see Section 5. It is well-known that the codimension of a distinguished variety cannot exceed the number m, see, e.g., [13, Proposition 2.6], and thus

$$c_\infty \leq \min(m, n).$$

We let Z_f denote the zero variety of $\mathcal{J}_f$ in X_{red}.

Let $\operatorname{reg} X$ denote the so-called (*Castelnuovo-Mumford*) *regularity* of $X \subset \mathbb{P}^N$, see Section 2.2 below. We can now formulate the main result of this paper.

THEOREM 1.2 (Main Theorem). *Assume that V is an algebraic subvariety of $\mathbb{C}^N$ of pure dimension n and let X be its closure in $\mathbb{P}^N$. There is a finite set of holomorphic differential operators L_α on $\mathbb{C}^N$ with polynomial coefficients and a number ν so that the following holds:*

(i) *for each point $x \in V$ the germs of L_α are Noetherian operators at x such that the conclusion in Theorem 1.1 holds,*

(ii) *if $F_1, \dots, F_m$ are polynomials of degree $\leq d$, Φ is a polynomial, and*

$$|L_\alpha \Phi|/|F|^\nu \text{ is locally bounded on } V_{\text{red}} \tag{1.4}$$

for each α, then there are polynomials $Q_1, \dots, Q_m$ such that (1.3) *holds and*

$$\deg(F_j Q_j) \\ \leq \max\big(\deg \Phi + \nu d^{c_\infty} \deg X_{\text{red}}, (d-1)\min(m, n+1) + \operatorname{reg} X\big). \tag{1.5}$$

If there are no distinguished varieties of $\mathcal{J}_f$ contained in $X_{\text{red},\infty}$, then d^{c_∞} shall be interpreted as 0.

In case V is reduced we can choose L_α as just the identity; then (ii) is precisely (part (i) of) Theorem A in [8]. If $V = \mathbb{C}^n$ we get back Hickel's theorem [17] mentioned above.

EXAMPLE 1.3. If we apply Theorem 1.2 to Nullstellensatz data, i.e., F_j with no common zeros on V and $\Phi = 1$, then the hypothesis (1.4) is fulfilled, and we thus get Q_j such that $F_1 Q_1 + \cdots + F_m Q_m - 1$ belongs to J_V and

$$\deg(F_j Q_j) \le \max\bigl(\nu d^{c_\infty} \deg X_{\mathrm{red}}, (d-1)\min(m, n+1) + \operatorname{reg} X\bigr).$$

See [8, Section 1] for a discussion of this estimate in the reduced case.

EXAMPLE 1.4. If f_j have no common zeros on X and Φ is any polynomial, then there is a solution to (1.3) such that

$$\deg F_j Q_j \le \max(\deg \Phi, (d-1)(n+1) + \operatorname{reg} X).$$

If $X = \mathbb{P}^n$, then $\operatorname{reg} X = 1$ and so we get back the classical Macaulay theorem.

REMARK 1.5. It follows that L_α is a set of Noetherian operators such that a polynomial $\Phi \in \mathbb{C}[x_1, \ldots, x_N]$ is in $J_V \subset \mathbb{C}[x_1, \ldots, x_N]$ if and only $L_\alpha \Phi = 0$ on V_{red} for each α. The existence of such a set is well-known, and a key point in the celebrated Ehrenpreis-Palamodov fundamental theorem, [12] and [24]; see also, e.g., [9] and [23].

REMARK 1.6. It turns out, see Theorem 4.1 below, that the Noetherian operators L_α in Theorem 1.2 have the following additional property: for each α there is a finite set of holomorphic differential operators $M_{\alpha,\gamma}$ such that

$$L_\alpha(\Phi\Psi) = \sum_\gamma L_\gamma \Phi \mathscr{M}_{\alpha,\gamma} \Psi$$

for any holomorphic functions Φ and Ψ. This formula shows that set of functions that satisfy (1.2) at a point x is indeed an ideal.

By homogenization, this kind of effective results can be reformulated as geometric statements: let $z = (z_0, \ldots, z_N)$, $z' = (z_1, \ldots, z_N)$, let $f_i(z) := z_0^d F_i(z'/z_0)$ be the d-homogenizations of F_i, considered as sections of $\mathscr{O}(d) \to \mathbb{P}^N$, and let $\varphi(z) := z_0^{\deg \Phi} \Phi(z'/z_0)$. Then there is a representation (1.3) on V with $\deg(F_j Q_j) \le \rho$ if and only if there are sections q_i of $\mathscr{O}(\rho - d)$ on $\mathbb{P}^N$ such that

$$f_1 q_1 + \cdots + f_m q_m = z_0^{\rho - \deg \Phi} \varphi$$

on X in $\mathbb{P}^N$; that is, the difference of the right and the left hand sides belongs to the sheaf $\mathscr{J}_X$.

To prove Theorem 1.2 we first have to define a suitable set of global Noetherian operators on $\mathbb{P}^N$. This is done in Section 4 following the ideas of Björk [10] in the local case, starting from a representation of $\mathscr{J}_X$ as the annihilator of a tuple of so-called Coleff-Herrera currents on $\mathbb{P}^N$. The rest of the proof of Theorem 1.2, given in Section 5, follows to a large extent the proof of Theorem A in [8]. By the construction in [5] we have a residue current R^X associated with $\mathcal{J}_X$ such that the annihilator ideal of R^X is precisely $\mathcal{J}_X$. Following the ideas in [8] we then form the "product" $R^f \wedge R^X$, where R^f is the current of Bochner-Martinelli type introduced in [1], inspired by [25]. By computations as in [27], the condition (1.4) ensures that ϕ annihilates this current at each point $x \in V_{\mathrm{red}}$. If ρ is large enough, this is reflected by the first entry of the right hand side of (1.5), then a geometric estimate from [13] ensures that the ρ-homogenization ϕ of Φ indeed satisfies a condition like (1.4) even at infinity. Therefore ϕ annihilates the current $R^f \wedge R^X$ everywhere on $\mathbb{P}^N$. For this argument it is important that the Noetherian operators extend to $\mathbb{P}^N$. The proof of Theorem 1.2 is then concluded along the same lines as in [8] by solving a sequence of $\bar\partial$-equations. If ρ is large enough, this is reflected by the second entry in the right hand side of (1.5), there are no cohomological obstructions. We then get a global representation of ϕ as a member of $\mathcal{O}(\rho) \otimes (\mathcal{J}_f + \mathcal{J}_X)$. After dehomogenization we get the desired representation (1.3).

In Section 2 we collect some necessary background material. In Section 3 we discuss global Coleff-Herrera currents on projective space. As mentioned above, the proof of our main theorem is given in the last two sections.

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2. Preliminaries

In this section we collect various definitions and facts that will be used later on.

2.1. Non-reduced analytic space

A reduced analytic space Z is locally described as an analytic subset of some open set $\Omega \subset \mathbb{C}^N$, and the sheaf $\mathcal{O}_Z$ of holomorphic functions on Z, the *structure sheaf*, is then isomorphic to $\mathcal{O}_\Omega/\mathcal{J}_Z$, where $\mathcal{J}_Z$ is the ideal sheaf of functions in Ω that vanish on Z. A non-reduced analytic space X (also referred to as an analytic scheme) with underlying reduced space Z and *structure sheaf* $\mathcal{O}_X$ is locally of the form $\mathcal{O}_X = \mathcal{O}_\Omega/\mathcal{J}$, where $\mathcal{J} \subset \mathcal{J}_Z$ is a coherent ideal sheaf with common zero set Z. Thus $\mathcal{J}_Z = \sqrt{\mathcal{J}}$ and $\mathcal{O}_Z$ is obtained from $\mathcal{O}_X$ by taking the quotient by all nilpotent elements in $\mathcal{O}_X$. Given the non-reduced space X we denote the underlying reduced space by X_{red}.

The space X has *pure dimension n* if for each $x \in X_{\mathrm{red}}$, all the associated prime ideals of the local ring $\mathcal{O}_x$ has dimension n. In particular, then X_{red} has pure dimension n.

2.2. Algebraic and projective varieties

We will only be concerned with analytic spaces that are globally embedded in some $\mathbb{C}^N$ or $\mathbb{P}^N$. An analytic subvariety $V \subset \mathbb{C}^N$ is *algebraic* if the sheaf $\mathcal{J}_V$ is generated by a finite number of polynomials. Let J_V be the corresponding ideal in the polynomial ring $\mathbb{C}[x_1, \dots, x_N]$. Let J_X be the homogeneous ideal in the graded ring $\mathbb{C}[x_0, \dots, x_N]$ generated by homogenizations of the elements in J_V. If J_V has pure dimension n, then J_X has pure dimension $n + 1$. In particular, 0 is not an associated prime ideal. Each homogeneous polynomial corresponds to a global section of the line bundle $\mathcal{O}(\ell) \to \mathbb{P}^N$ for some ℓ. These sections define a coherent analytic sheaf $\mathcal{J}_X$ over $\mathbb{P}^N$ of pure dimension n. We define the closure X of V as the analytic subvariety of $\mathbb{P}^N$ with structure sheaf $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^N}/\mathcal{J}_X$. It is clear that the sheaf $\mathcal{J}_X$ coincides with the sheaf $\mathcal{J}_V$ defined by the ideal J_V in $\mathbb{C}^N$.

Let S be the graded ring $\mathbb{C}[x_0, \dots, x_N]$ and let $S(-d)$ be the S-module that is equal to S but with the gradings shifted by d. Let J_X be the homogeneous ideal in S of all forms that belong to $\mathcal{J}_X$. Since 0 is not an associated prime ideal of J_X, cf. [14, Corollary 20.14], see also [8, Section 2.7], there is a graded free resolution

$$0 \longrightarrow \bigoplus_{i=1}^{r_N} S(-d_N^i) \xrightarrow{c_N} \cdots \xrightarrow{c_2} \bigoplus_{i=1}^{r_1} S(-d_1^i) \xrightarrow{c_1} S \longrightarrow S/J_X \longrightarrow 0 \tag{2.1}$$

of the S-module S/J_X, where $c_k = (c_k^{ij})$ are matrices of homogeneous forms in $\mathbb{C}^{N+1}$ with $\deg c_k^{ij} = d_k^j - d_{k-1}^i$. The number

$$\operatorname{reg} X := \max_{k,i}(d_k^i - k) + 1$$

is called the Castenouvo-Mumford regularity of X in $\mathbb{P}^N$, see, e.g., [15]. This number describes the complexity of the embedding of X in $\mathbb{P}^N$; thus two isomorphic analytic spaces embedded in different ways may have different regularities.

2.3. Some residue theory

Let Y be a (smooth) complex manifold of dimension N. Given a holomorphic function f on Y, following Herrera and Lieberman [16], one can define the principal value current $1/f$ as the limit

$$\lim_{\epsilon \to 0} \chi(|f|^2 v/\epsilon)\frac{1}{f},$$

where $\chi(t)$ is the characteristic function of the interval $[1, \infty)$ or a smooth approximand and v is any smooth strictly positive function. The existence of this limit for a general f relies on Hironaka's theorem that ensures that there is a modification $\pi\colon \widetilde{Y} \to Y$ such that $\pi^* f$ is locally a monomial. It is readily checked that $f(1/f) = 1$ and $f\bar\partial(1/f) = 0$. The current $1/f$ is well-defined even if f is a holomorphic section of a Hermitian line bundle over Y, since $a(1/af) = 1/f$ if a is holomorphic and nonvanishing.

EXAMPLE 2.1. In one complex variable it is quite elementary to see that the principal value current $1/s^{m+1}$ exists and that

$$\bar\partial \frac{1}{s^{m+1}} \wedge ds.\xi = \frac{2\pi i}{m!} \frac{\partial^m}{\partial s^m} \xi(0),$$

for test functions ξ.

The sheaf $\mathcal{PM} = \mathcal{PM}_Y$ of *pseudomeromorphic currents*, introduced in [6], [3], consists of currents on Y that are finite sums of direct images under (compositions of) modifications, simple projections and open inclusions of currents of the form

$$\frac{\xi}{s_1^{\alpha_1} \dots s_{\ell-1}^{\alpha_{\ell-1}}} \wedge \bar\partial \frac{1}{s_\ell^{\alpha_\ell}} \wedge \cdots \wedge \bar\partial \frac{1}{s_m^{\alpha_m}}, \qquad m \le n,$$

in some $\mathbb{C}^m_s$ and ξ is a smooth form with compact support.

The sheaf $\mathcal{PM}$ is closed under $\bar\partial$ (and ∂) and multiplication by smooth forms. If τ is in $\mathcal{PM}$ and has support on an analytic subset $V \subset Y$ and η is a holomorphic form that vanishes on V, then

$$\overline{\eta} \wedge \tau = 0, \qquad d\overline{\eta} \wedge \tau = 0. \tag{2.2}$$

The first equality roughly speaking means that τ does not involve anti-holomorphic derivatives. By a standard argument the second equality in (2.2) implies:

Dimension principle: If τ is a pseudomeromorphic current on Y of bidegree $(*, p)$ that has support on an analytic subset V of codimension $> p$, then $\tau = 0$.

Let $\mathcal{U} \subset Y$ be an open subset. If τ is in $\mathcal{PM}(\mathcal{U})$ and $V \subset \mathcal{U}$ is an analytic subvariety, then the natural restriction of τ to the open set $\mathcal{U} \setminus V$ has a canonical extension as a principal value to a pseudomeromorphic current $\mathbf{1}_{\mathcal{U}\setminus V}\tau$ on $\mathcal{U}$. If h is a holomorphic tuple in $\mathcal{U}$ with common zero set V, and χ is a smooth approximand χ of the characteristic function of the interval $[1, \infty)$, then

$$\mathbf{1}_{\mathcal{U}\setminus V}\tau = \lim_{\epsilon\to 0} \chi(|h|^2/\epsilon)\tau. \tag{2.3}$$

It follows that $\mathbf{1}_V\tau := \tau - \mathbf{1}_{\mathscr{U}\setminus V}\tau$ is pseudomeromorphic in $\mathscr{U}$ and has support on V. Notice that if α is a smooth form, then $\mathbf{1}_V\alpha\wedge\tau = \alpha\wedge\mathbf{1}_V\tau$. Moreover, if $\pi\colon\widetilde{\mathscr{U}}\to\mathscr{U}$ is a modification, $\tilde\tau$ is in $\mathscr{PM}(\widetilde{\mathscr{U}})$, and $\tau=\pi_*\tilde\tau$, then

$$\mathbf{1}_V\tau = \pi_*\big(\mathbf{1}_{\pi^{-1}V}\tilde\tau\big)$$

for any analytic set $V\subset\mathscr{U}$. For any analytic sets W, $W'\subset\mathscr{U}$,

$$\mathbf{1}_W\mathbf{1}_{W'}\tau = \mathbf{1}_{W\cap W'}\tau.$$

Let $Z\subset Y$ be an analytic subset of pure codimension p and let τ be a pseudomeromorphic current of bidegree $(N,*)$ with support on Z. We say that τ has the *standard extension property*, SEP, with respect to Z if $\mathbf{1}_V\tau=0$ for each subvariety $V\subset Z\cap\mathscr{U}$ of positive codimension, where $\mathscr{U}\subset Y$ is some open subset. The sheaf of such currents is denoted by $\mathscr{W}^Z$. If $Z=Y$ we write $\mathscr{W}$ rather than $\mathscr{W}^Y$. The subsheaf of $\mathscr{W}^Z$ of $\bar\partial$-closed currents of bidegree (N,p) is called the sheaf of Coleff-Herrera currents[1], $\mathscr{CH}^Z$, on Z.

REMARK 2.2. The sheaf $\mathscr{CH}^Z$ was introduced by Björk, in a slightly different way. For the equivalence to the definition given here, see [2, Section 5].

EXAMPLE 2.3. Let $[Z]$ be the Lelong current associated with Z and let β be a smooth form of bidegree $(p,*)$. Then $\mu=\beta\wedge[Z]$ is in $\mathscr{W}^Z$. If β is holomorphic, then μ is in $\mathscr{CH}^Z$. See, e.g., [2, Example 4.2].

PROPOSITION 2.4. *If $\mathscr{L}$ is a holomorphic differential operator and τ is in $\mathscr{W}^Z$, then $\xi\mapsto\tau.\mathscr{L}\xi$ defines a current in $\mathscr{W}^Z$.*

PROOF. It is a local statement so by induction it is enough to let $\mathscr{L}$ be a partial derivative $\partial/\partial\zeta_1$ with respect to some local coordinate system. Let L denote the Lie derivative with respect to this vector field. Since ξ has bidegree $(0,*)$, $(\partial/\partial\zeta_1)\xi = L\xi$. Thus

$$\tau.(\partial/\partial\zeta_1)\xi = \tau.L\xi = \pm L\tau.\xi,$$

and $L\tau$ is in $\mathscr{W}^Z$ according to [7, Theorem 3.7].

2.4. Almost semi-meromorphic currents

We say that a current b on a smooth manifold Y is *almost semi-meromorphic*, $b\in ASM(Y)$, if there is a modification $\pi\colon Y'\to Y$, a holomorphic generically non-vanishing section σ of a line bundle $L\to Y'$ and an L-valued smooth form ω such that

$$b=\pi_*\frac{\omega}{\sigma}, \tag{2.4}$$

[1] We adopt here the convention from [10]; in, e.g., [27] these currents have bidegree $(0,p)$.

where ω/σ denotes the principal value current. This class of currents was introduced in [3] and studied in more detail in [7]. All results in this subsection can be found in the latter reference.

Let $ZSS(b)$, the Zariski singular support of b, be the smallest analytic set such that b is smooth in the complement.

We will need the following results.

PROPOSITION 2.5 ([7], Theorem 4.25). *If b is almost semi-meromorphic on Y and $\mathscr{L}$ is a holomorphic differential operator, then $\mathscr{L}b$ is almost semi-meromorphic as well.*

Clearly, $ZSS(\mathscr{L}b) \subset ZSS(b)$.

THEOREM 2.6 ([7], Theorem 4.8). *If $b \in ASM(Y)$ and τ is any pseudomeromorphic current in Y, then there is a unique current T in Y that coincides with $b \wedge \tau$ outside $ZSS(b)$ and such that $\mathbf{1}_{ZSS(b)}T = 0$.*

We will denote the extension T by $b \wedge \tau$ as well. It follows from (2.3) that

$$b \wedge \tau = \lim_{\delta} \chi_\delta b \wedge \tau$$

if $\chi_\delta = \chi(|g|^2/\delta)$ where g is a holomorphic tuple whose zero set is precisely $ZSS(b)$. It is not hard to check, cf. [7, Proposition 4.9], that if V is any analytic set, then

$$\mathbf{1}_V(b \wedge \tau) = b \wedge \mathbf{1}_V \tau. \tag{2.5}$$

It follows from (2.5) that $b \in ASM(Y)$ induces a mapping

$$\mathscr{W}^Z \to \mathscr{W}^Z, \quad \tau \mapsto b \wedge \tau.$$

Given $a \in ASM(Y)$ and $\tau \in \mathscr{PM}^Y$ we define

$$\bar\partial a \wedge \tau := \bar\partial(a \wedge \tau) - (-1)^{\deg a} a \wedge \bar\partial\tau$$

The definition is made so that the formal Leibniz rule holds.

REMARK 2.7. Clearly $\bar\partial a = b + r(a)$ where $b = \mathbf{1}_{X\setminus ZSS(a)}\bar\partial a$ and $r(a)$, the residue of a, has support on $ZSS(a)$. One can check, cf. [7, Proposition 4.16], that in fact $b \in ASM(X)$. Thus we can define $r(a) \wedge \tau := \bar\partial a \wedge \tau - b \wedge a$. If χ_δ is as above, then

$$r(a) \wedge \tau = \lim_{\delta} \bar\partial\chi_\delta \wedge a \wedge \tau. \tag{2.6}$$

If a is holomorphic outside $ZSS(a)$, then clearly the support of $\bar\partial a \wedge \tau$ is contained in $\operatorname{supp} \tau \cap ZSS(a)$. In particular, if $\gamma_1, \dots, \gamma_p$ are holomorphic functions, then by induction we can form the current

$$\bar\partial\frac{1}{\gamma_p}\wedge\cdots\wedge\bar\partial\frac{1}{\gamma_1}. \tag{2.7}$$

Clearly it is $\bar\partial$-closed and has support on $Z_\gamma = \{\gamma_1 = \cdots = \gamma_p = 0\}$. If in addition Z_γ has codimension p, then (2.7) is anti-commuting in its factors, see, e.g., [6, Section 2]. In this case we call it the *Coleff-Herrera product* μ^γ formed by the γ_j. It is well-known, and was first proved by Dickenstein-Sessa and Passare, that the annihilator $\mathrm{ann}\,\mu^\gamma = \{\phi \in \mathscr{O} : \phi\mu^\gamma = 0\}$ is precisely equal to the ideal (γ) generated by $\gamma_1, \ldots, \gamma_p$, see, [2, Eq. (4.3)] for the setting used here. It follows by the dimension principle that μ^γ is in $\mathscr{W}^{Z_\gamma}$. If ω is a holomorphic $(N, 0)$-form, therefore $\mu^\gamma \wedge \omega$ is in $\mathscr{CH}^{Z_\gamma}$.

Any Coleff-Herrera current μ can be written locally as $\mu = a\mu^\gamma \wedge \omega$ for such a tuple γ and some holomorphic function a, see, e.g., [2, Theorem 1.1]. Thus the annihilator $\mathrm{ann}\,\mu$ is the kernel of the sheaf mapping $\mathscr{O} \to \mathscr{O}/(\gamma)$, $\phi \mapsto a\phi$, and hence $\mathrm{ann}\,\mu$ is coherent.

Let $S \to Y$ be a vector bundle. We say that $b \in ASM(Y, S)$ if there is a representation (2.4), where ω is a smooth section of $L \otimes \pi^* S$. The statements above have analogues for S-valued sections. For instance, if S is a line bundle and $\gamma_j \in \mathscr{O}(Y, S)$, then (2.7) is an S^{-p}-valued current.

3. Global Coleff-Herrera currents on $\mathbb{P}^N$

Let δ_x be interior multiplication by the vector field

$$\sum_1^N x_j \frac{\partial}{\partial x_j}$$

on $\mathbb{C}^{N+1}$ and recall that a differential form ξ on $\mathbb{C}^{N+1} \setminus \{0\}$ is projective, i.e., the pullback of a form on $\mathbb{P}^N$, if and only if $\delta_x\xi = \delta_{\bar x}\xi = 0$, where $\delta_{\bar x}$ is the conjugate of δ_x. We will identify forms on $\mathbb{P}^N$ and projective forms. Notice that

$$\Omega = \delta_x(dx_0 \wedge \cdots \wedge dx_N)$$

is a non-vanishing section of the trivial bundle over $\mathbb{P}^N$, realized as a $(N, 0)$-form on $\mathbb{P}^N$ with values in $\mathscr{O}(N + 1)$.

Let $\gamma_1, \ldots, \gamma_p$ be holomorphic sections of $\mathscr{O}(r)$ such that their common zero set Z_γ has codimension p. Then, cf. Section 2.4 above,

$$\mu^\gamma \wedge \Omega = \bar\partial\frac{1}{\gamma_p}\wedge\cdots\wedge\bar\partial\frac{1}{\gamma_1}\wedge\Omega$$

is a global section of $\mathcal{CH}^{Z_\gamma} \otimes \mathcal{O}(-pr + N + 1)$.

LEMMA 3.1. *Let $Z \subset Z_\gamma$ be a reduced projective variety of pure codimension p and let μ be a global section of $\mathcal{CH}^Z \otimes \mathcal{O}(\ell + N + 1)$ such that*

$$\gamma_1\mu = \cdots = \gamma_p\mu = 0. \tag{3.1}$$

If $p \le N - 1$, then there is a global holomorphic section a of $\mathcal{O}(\ell + pr)$ such that

$$\mu = a\bar\partial\frac{1}{\gamma_p} \wedge \cdots \wedge \bar\partial\frac{1}{\gamma_1} \wedge \Omega.$$

If $p = N$ and $\ell + N \ge 0$, then the same conclusion holds.

In particular we see that if $p \le N - 1$ and $\ell + pr < 0$, then $\mu = 0$.

PROOF. Let us introduce a trivial vector bundle E of rank p with global holomorphic frame elements $e_1, \ldots, e_p$ and let $e_1^*, \ldots, e_p^*$ be the dual frame for E^*. We then have the mapping interior multiplication $\delta_\gamma\colon \Lambda^{*+1}E \to \Lambda^* E$ by the section $\gamma := \gamma_1 e_1^* + \cdots + \gamma_p e_p^*$ of E^*. We consider the exterior algebra of $E \oplus T^*\mathbb{P}^N$ so that $d\bar x_j \wedge e_j = -e_j \wedge d\bar x_j$ etc. Then both δ_γ and $\bar\partial$ extend to mappings on currents with values in ΛE, and

$$\delta_\gamma\bar\partial = -\bar\partial\delta_\gamma. \tag{3.2}$$

Let $e = e_1 \wedge \cdots \wedge e_p$. Recall that $H^{N,k}(\mathbb{P}^N, \mathcal{O}(\nu)) = 0$ if either $1 \le k \le N-1$ or $k = N$ and $\nu \ge 1$; see, e.g., [11, Ch. VII, Theorem 10.7]. If $p \le N - 1$, or $\ell + N + 1 \ge 1$, we can therefore find a global solution to $\bar\partial w_{p-1} = \mu \wedge e$. In view of (3.2) and (3.1) we have that

$$\bar\partial\delta_\gamma w_{p-1} = -\delta_\gamma\bar\partial w_{p-1} = -\delta_\gamma(\mu \wedge e) = 0.$$

Thus we can successively solve

$$\bar\partial w_{p-1} = \mu \wedge e, \quad \bar\partial w_{p-2} = \delta_\gamma w_{p-1}, \quad \ldots, \quad \bar\partial w_0 = \delta_\gamma w_1.$$

Then $a \wedge \Omega := \delta_\gamma w_0$ is a $\bar\partial$-closed, and thus a holomorphic, $(N, 0)$-form with values in $\mathcal{O}(\ell + pr + N + 1)$. Altogether,

$$(\delta_\gamma - \bar\partial)w = a \wedge \Omega - \mu \wedge e$$

if $w = w_0 + \cdots + w_{p-1}$. As in [2, Examples 3.1 or 3.2] we can find a global current U such that

$$(\delta_\gamma - \bar\partial)U = 1 - \mu^\gamma \wedge e.$$

Thus

$$(\delta_\gamma - \bar\partial)(aU \wedge \Omega - w) = \mu - a\mu^\gamma \wedge \Omega.$$

Since the right hand side is in $\mathcal{CH}^Z$ it now follows from [2, Theorem 3.3] that it must vanish.

EXAMPLE 3.2. Given a global section μ of $\mathcal{CH}_Z \otimes \mathcal{O}(\ell)$ one can always find γ_j such that (3.1) holds. In fact, for a large enough r_0 there are sections $g'_1, \dots, g'_m$ of $\mathcal{O}(r_0)$ that generate the ideal sheaf $\mathcal{J}_Z \subset \mathcal{O}_{\mathbb{P}^N}$. If $g_1, \dots, g_p$ are generic linear combinations of the g'_j, then $Z_g = \{g_1 = \cdots = g_p = 0\}$ has codimension p, $Z_g \supset Z$, and (expressed in a local frame) $dg_1 \wedge \cdots \wedge dg_p \neq 0$ on Z_{reg}. If $\gamma_j = g_j^{\mathfrak{m}_j+1}$ and $\mathfrak{m}_j$ are large enough, then (3.1) holds.

4. Björk-type representation of global Coleff-Herrera currents

In this section we express the action $\mu.\xi$ of a global Coleff-Herrera current μ on a test form ξ as an integral over Z of $\mathcal{M}\xi$, where $\mathcal{M}$ is a certain differential operator.

As usual we identify smooth sections ψ of the line bundle $\mathcal{O}(\ell)$ by ℓ-homogeneous smooth functions on $\mathbb{C}^{N+1} \setminus \{0\}$. Notice that then each $\partial/\partial x_j$, $j = 0, \dots, N$, induces a differential operator $\mathcal{O}(\ell) \to \mathcal{O}(\ell - 1)$. We say that a finite sum

$$\mathcal{L} = \sum_\alpha v_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$$

is a holomorphic differential operator on $\mathbb{P}^N$ of *degree* r if the coefficients v_α are holomorphic sections of $\mathcal{O}(r + |\alpha|)$. Such an $\mathcal{L}$ maps $\mathcal{O}(\ell) \to \mathcal{O}(\ell + r)$ for each ℓ. The *order* of $\mathcal{L}$ is the maximal occurring $|\alpha|$ as usual.

Consider the affinization $\mathbb{C}^N \simeq \{x_0 \neq 0\}$. Notice that there is a one-to-one correspondence between smooth sections of $\mathcal{O}(\ell)$ over $\mathbb{C}^N$ and smooth functions in $\mathbb{C}^N$, via the frame $[x_0, \dots, x_N] \mapsto x_0^\ell$ for $\mathcal{O}(\ell)$ over $\mathbb{C}^N$. More concretely, given the section ϕ one gets the associated function by just letting $x_0 = 1$. Conversely, given Φ, then $\phi(x) = x_0^\ell \Phi(x'/x_0)$. In this way a differential operator of degree r gives rise to a differential operator

$$L = \sum_{|\alpha'| \leq M} V_{\alpha'}(x') \frac{\partial^{\alpha'}}{\partial x^{\alpha'}}$$

where $V_{\alpha'}(x')$ are polynomials of degree at most $r + |\alpha'|$. Notice however, that the resulting affine L will depend on ℓ unless $\mathcal{L}(x_0\phi) = x_0\mathcal{L}\phi$ for all ϕ. For instance, the differential operator $\mathcal{L} = \partial/\partial x_0$, that has order 1 and degree -1, induces

$$L = \ell - \sum_{j=1}^N x_j \frac{\partial^j}{\partial x^j}.$$

Notice that $\mathscr{L}$, as well as an associated affine differential operator L, act on smooth $(0,*)$-forms as well.

The following statement is a global version of a construction due to Björk, [10]. A similar result is obtained in [28, Theorem 4.2].

THEOREM 4.1. *Assume that $Z \subset \mathbb{P}^N$ has pure codimension p, that μ is a global section of $\mathcal{CH}_Z \otimes \mathcal{O}(r)$, and assume that $p \le N-1$ or $r+1 \ge 0$. Let $\mathcal{J} = \operatorname{ann}\mu$. There is a multiindex $\mathfrak{m} = (\mathfrak{m}_1, \ldots, \mathfrak{m}_p)$, a number ρ, and for each $\alpha \le \mathfrak{m}$ there are holomorphic differential operators $\mathscr{L}_\alpha$ and $\mathscr{M}_{\mathfrak{m}-\alpha}$, such that* $\deg \mathscr{L}_\alpha + \deg \mathscr{M}_{\mathfrak{m}-\alpha} = \rho$*, and a global meromorphic $(n,0)$-form τ with values in $\mathcal{O}(-\rho)$, not identically polar on any irreducible component of Z, such that the following hold:*

(i) *for any global holomorphic section ϕ of $\mathcal{O}(\ell)$ and any test form ξ of bidegree $(0,n)$ with values in $\mathcal{O}(-r-\ell)$, we have*

$$\phi\mu.\xi = \sum_{\alpha \le m} \int_Z \tau \wedge \mathscr{L}_\alpha \phi \wedge \mathscr{M}_{\mathfrak{m}-\alpha}\xi, \tag{4.1}$$

(ii) *for each point $x \in Z$, a germ $\psi \in \mathcal{O}_x$ is in $\mathcal{J}_x$ if and only if*

$$\mathscr{L}_\alpha \psi \in \sqrt{\mathcal{J}_x}, \quad \alpha \le \mathfrak{m},$$

(iii) *for each $\alpha \le \mathfrak{m}$ there are holomorphic differential operators $\mathscr{M}_{\alpha,\gamma}$, $\gamma \le \alpha$, such that*

$$\mathscr{L}_\alpha(\phi\psi) = \sum_{\gamma \le \alpha} \mathscr{L}_\gamma \phi \mathscr{M}_{\alpha,\gamma}\psi$$

for all holomorphic sections ϕ and ψ of $\mathcal{O}(\ell)$ and $\mathcal{O}(\ell')$.

PROOF. To begin with we choose $g_1, \ldots, g_p$, $\mathfrak{m} := (\mathfrak{m}_1, \ldots, \mathfrak{m}_p)$, and a as in Example 3.2 and Lemma 3.1 so that

$$\mu = a\mu^{g^{\mathfrak{m}+\mathbf{1}}} \wedge \Omega. \tag{4.2}$$

After a projective transformation on $\mathbb{P}^N$, i.e., a linear change of variables on $\mathbb{C}^{N+1}$, we may assume that each irreducible component of Z intersects the affine space $\mathbb{C}^N := \{x_0 \ne 0\}$. Then the affinizations G_j of g_j are polynomials in $\mathbb{C}^N$ such that $dG_1 \wedge \cdots \wedge dG_p$ is nonvanishing on $Z_{\text{reg}} \cap \mathbb{C}^N$, cf. Example 3.2. Let $x' = (x_1, \ldots, x_N)$. After possibly a linear transformation of $\mathbb{C}^N$, we may assume that the polynomial

$$H := \det \frac{\partial G}{\partial \eta}$$

is generically nonvanishing on $Z \cap \mathbb{C}^N$, where

$$x' = (\zeta, \eta) = (\zeta_1, \dots, \zeta_n, \eta_1, \dots \eta_p).$$

Let us introduce the short hand notation

$$\bar\partial \frac{1}{G^{\mathfrak{m}+\mathbf{1}}} = \bar\partial \frac{1}{G_1^{\mathfrak{m}_1+1}} \wedge \cdots \wedge \bar\partial \frac{1}{G_p^{\mathfrak{m}_p+1}}.$$

We first look for a representation of the Coleff-Herrera current

$$\tilde\mu = \bar\partial \frac{1}{G^{\mathfrak{m}+\mathbf{1}}} \wedge d\eta \wedge d\zeta$$

at points x on $Z' := Z \cap \mathbb{C}^N \cap \{H \neq 0\}$. Locally at such a point we can make the change of variables

$$w = G(\zeta, \eta), \quad z = \zeta.$$

If Ξ is a smooth $(0, n)$-form with small support, and Φ is holomorphic, with the notation $m! = m_1! \dots m_p!$ and $\partial_w^\alpha = \partial^{|\alpha|}/\partial w^\alpha$, etc, in view of Example 2.1 we then have

$$\begin{aligned}
\Phi\tilde\mu.\Xi &= \int \bar\partial \frac{1}{G^{\mathfrak{m}+\mathbf{1}}} \wedge d\eta \wedge d\zeta \wedge \Phi\Xi \\
&= \pm \int \bar\partial \frac{1}{w^{\mathfrak{m}+\mathbf{1}}} \wedge dw \wedge dz \wedge \frac{\Xi}{H}\Phi \\
&= \pm \int_{w=0} \frac{(2\pi i)^p}{\mathfrak{m}!} dz \wedge \partial_w^{\mathfrak{m}} \left(\frac{\Xi}{H} \Phi \right) \\
&= \pm \sum_{\alpha \le \mathfrak{m}} \int_{w=0} \frac{(2\pi i)^p}{(\mathfrak{m}-\alpha)!\,\alpha!} dz \wedge \partial_w^{\mathfrak{m}-\alpha} \left(\frac{\Xi}{H} \right) \partial_w^\alpha \Phi.
\end{aligned}$$

Now, notice that

$$\partial_\eta = (\partial_\eta G)\partial_w$$

so that

$$\partial_w = \frac{\Gamma}{H}\partial_\eta,$$

where Γ is a matrix of polynomials. It is readily checked that

$$\tilde L_\alpha := H^{2|\alpha|} \left(\frac{\Gamma}{H} \partial_\eta \right)^\alpha$$

has a holomorphic extension across $H = 0$. Let us define

$$M_\beta \Xi = \pm \frac{(2\pi i)^p}{\beta!\,(\mathfrak{m}-\beta)!} H^{1+|\mathfrak{m}|+2|\beta|} \left(\frac{\Gamma}{H}\partial_\eta\right)^\beta \frac{\Xi}{H}.$$

Then also M_β is holomorphic across $H = 0$.

With $T = dz = d\zeta$, we have that

$$\Phi\tilde{\mu}.\Xi = \int_{Z'} \sum_{\alpha\le\mathfrak{m}} \frac{T}{H^{3|\mathfrak{m}|+1}} \wedge M_{\mathfrak{m}-\alpha}\Xi \wedge \tilde{L}_\alpha \Phi$$

for Ξ with support close to x. We claim that if Φ is a germ of a holomorphic function at x, then $\Phi\tilde{\mu}_x = 0$ if and only if $\tilde{L}_\alpha\Phi = 0$ on Z_x for all $\alpha \le \mathfrak{m}$. In fact,

$$\Phi\tilde{\mu}_x = 0 \iff \Phi\bar\partial \frac{1}{G^{\mathfrak{m}+1}}\bigg|_x = 0 \iff \Phi\bar\partial\frac{1}{w^{\mathfrak{m}+1}}\bigg|_x = 0$$
$$\iff \partial_w^\alpha \Phi = 0 \text{ on } Z_x,\ \alpha \le \mathfrak{m} \iff \tilde{L}_\alpha \Phi = 0 \text{ on } Z_x,\ \alpha\le\mathfrak{m}. \quad (4.3)$$

Now, for each $\alpha \le m$, let us homogenize the coefficients in $\tilde{L}_\alpha$ to obtain $\tilde{\mathscr{L}}_\alpha$ for some fixed degree, and then let us homogenize $M_{\mathfrak{m}-\alpha}$ to $\mathscr{M}_{\mathfrak{m}-\alpha}$ so that the sum of their degrees is a fixed number ρ. Let τ' be the homogenization of $T = d\zeta$, i.e.,

$$\tau' = d\frac{x_1}{x_0} \wedge \cdots \wedge d\frac{x_n}{x_0}$$

if $x = (x_0, \ldots, x_N) = (x_0, \zeta, \eta)$. Finally let us homogenize $H^{3|\mathfrak{m}|+1}$ to h so that $\tau := \tau'/h$ takes values in $\mathscr{O}(-\rho)$. We possibly get some factors x_0 in the denominator, but since Z has no irreducible component in $\{x_0 = 0\}$ this is acceptable.

Let us define the global current

$$\tilde{\mu} := \mathbf{1}_Z \mu^{g^{\mathfrak{m}+1}} \wedge \Omega \quad (4.4)$$

in $\mathbb{P}^N$. In view of (4.2) it takes values in $\mathscr{O}(r - \deg a)$. At each point $x \in Z'$ it is the $(r - \deg a)$-homogenization of our previous $\tilde{\mu}$ but the global current is not necessarily $\bar\partial$-closed at x_0. However, in view of (4.2), (2.5), and (4.4),

$$a\tilde{\mu} = a\mathbf{1}_Z\mu^{g^{\mathfrak{m}+1}} \wedge \Omega = \mathbf{1}_Z a\mu^{g^{\mathfrak{m}+1}} \wedge \Omega = \mathbf{1}_Z\mu = \mu,$$

since μ has support on Z, and thus $a\tilde{\mu}$ is $\bar\partial$-closed.

For holomorphic sections ϕ of $\mathscr{O}(\ell - \deg a)$ and test forms ξ of bidegree $(0, n)$ with support in $\mathbb{P}^N \setminus \{h = 0,\ x_0 = 0\}$ and values in $\mathscr{O}(-r-\ell)$ we have

$$\phi\tilde{\mu}.\xi = \int_Z \sum_{\alpha\le\mathfrak{m}} \tau \wedge \mathscr{M}_{\mathfrak{m}-\alpha}\xi \wedge \tilde{\mathscr{L}}_\alpha\phi. \tag{4.5}$$

By Theorem 2.6, $\tau \wedge \tilde{\mathscr{L}}\phi \wedge [Z]$ is a global section of $\mathscr{W}^Z \otimes \mathscr{O}(r+\ell)$ and thus the integrals on the right hand side of (4.5) exist as a principal values for any test form ξ. In view of Proposition 2.4 the right hand side of (4.5) defines the action on ξ of a global section of $\mathscr{W}^Z \otimes \mathscr{O}(r+\ell)$. Since $\{h = 0,\ x_0 = 0\} \cap Z$ has positive codimension on Z it follows by the SEP that the equality (4.5) holds for all ξ.

Define the holomorphic differential operators $\mathscr{L}_\alpha$ by the equality

$$\mathscr{L}_\alpha\phi = \tilde{\mathscr{L}}_\alpha(a\phi). \tag{4.6}$$

Then (4.1) follows from (4.5). Thus (i) is proved.

For $x \in Z' = Z \setminus \{h = 0,\ x_0 = 0\}$ we have, by (4.3) and (4.6), that

$$\phi\mu_x = 0 \quad \text{if and only if} \quad \mathscr{L}_\alpha\phi = 0 \text{ on } Z_x,\ \alpha \le \mathfrak{m}. \tag{4.7}$$

Again since $\{h = 0,\ x_0 = 0\} \cap Z$ has positive codimension on Z, it follows by continuity and the SEP that (4.7) holds for all $x \in Z$. Thus (ii) is proved.

To see (iii), just notice that

$$\tilde{L}_\alpha(\Phi\Psi) = \sum_{\gamma\le\alpha} L_\gamma\Phi c_{\alpha,\gamma} L_{\alpha-\gamma}\Phi,$$

where $c_{\alpha,\gamma}$ are binomial coefficients. After homogenization and replacing ϕ by $a\phi$ we get (iii) with $\mathscr{L}_{\alpha,\gamma} = c_{\alpha,\gamma}\mathscr{L}_{\alpha-\gamma}$.

REMARK 4.2. One can check, cf. [6, Section 5], that $\phi\mathbf{1}_Z\tilde{\mu} = 0$ if and only if ϕ is in the intersection of the primary ideals of $(g^{\mathfrak{m}+1})$ associated with the irreducible components of Z.

Let μ be a global section of $\mathscr{C}\mathscr{H}^Z \otimes \mathscr{O}(r)$ in $\mathbb{P}^N$ and let b be a global almost semi-meromorphic current of bidegree $(0, *)$ with values in $\mathscr{O}(r_1)$. Then $b\mu$ is a section of $\mathscr{W}^Z \otimes \mathscr{O}(r + r_1)$. Let us also assume that $ZSS(b) \cap Z$ has positive codimension in Z. Consider a representation of μ as in Theorem 4.1. In view of Theorem 2.5 we can define differential operators $\widehat{M}_\gamma$ with almost semi-meromorphic coefficients so that

$$\widehat{M}_\gamma\xi = M_\gamma(b\xi).$$

For test forms ξ of bidegree $(0, *)$ with values in $\mathscr{O}(-r-\ell)$ and with support outside $ZSS(b)$, and any global holomorphic section ϕ of $\mathscr{O}(\ell)$ we have

$$\phi b\mu.\xi = \sum_{\alpha\leq\mathfrak{m}} \int_Z \tau \wedge \mathscr{L}_\alpha\phi \wedge \widehat{M}_{\mathfrak{m}-\alpha}\xi. \tag{4.8}$$

In view of Propositions 2.6 and 2.4 the right hand side defines a global section of $\mathscr{W}^Z \otimes \mathscr{O}(r+r_1)$. Since $Z \cap ZSS(b)$ has positive codimension in Z, it follows that (4.8) holds globally.

5. Proof of Theorem 1.2

Let X be our non-reduced subspace of $\mathbb{P}^N$. As was mentioned in the introduction the proof relies on the global current $R^f \wedge R^X$ that we first discuss.

5.1. The current R^X

Given a vector bundle $E \to \mathbb{P}^N$, let $\mathscr{O}(E)$ denote the associated locally free analytic sheaf. We can find a locally free resolution

$$0 \longrightarrow \mathscr{O}(E_N) \xrightarrow{c_N} \cdots \xrightarrow{c_2} \mathscr{O}(E_1) \xrightarrow{c_1} \mathscr{O}(E_0) \longrightarrow \mathscr{O}_{\mathbb{P}^N}/\mathscr{J}_X \longrightarrow 0$$

of $\mathscr{O}_{\mathbb{P}^N}/\mathscr{J}_X$, where E_0 is a trivial line bundle and $E_k = \oplus_i^{r_k} \mathscr{O}(-d_k^i)$ for suitable positive numbers d_k^i, see, e.g., [8]. In fact, we can use the "same" mappings $c_k = (c_k^{ij})$ as in (2.1) but with c_k^{ij} considered as sections of $\mathscr{O}(d_k^j - d_{k-1}^i)$. There is a natural choice of Hermitian metrics on E_k and following [5, Sections 3 and 6] there is an associated current

$$R^X = R_p^X + \cdots + R_N^X$$

with support on X_{red}, where R_k^X are $(0, k)$-currents that take values in E_k, and with the property that $\phi R^X = 0$ if and only if $\phi \in \mathscr{J}_X$. Furthermore,

$$\bar\partial R_k^X = c_{k+1} R_{k+1}^X, \qquad k \geq 0. \tag{5.1}$$

PROPOSITION 5.1. *There is a bundle*

$$F = \bigoplus_{i=1}^{r_F} \mathscr{O}(d_F), \tag{5.2}$$

a global section μ of $\mathscr{CH}^{X_{\text{red}}} \otimes F \otimes \mathscr{O}(N+1)$, and an almost semi-meromorphic section b of $\text{Hom}(F, \oplus_{i=p}^{N+1} E_k)$ *such that*

$$R^X \wedge \Omega = b\mu \tag{5.3}$$

in $\mathbb{P}^N$.

Proof. Since the kernel $\mathscr{K}$ of $c^*_{p+1}\colon \mathscr{O}(E^*_p)\to\mathscr{O}(E^*_{p+1})$ is coherent, for a large enough integer d_F, $\mathscr{K}\otimes\mathscr{O}(d_F)$ is generated by global sections $g_1,\dots,g_{r_F}$. We therefore have a surjective sheaf mapping $\bigoplus_1^{r_F}\mathscr{O}\to\mathscr{K}\otimes\mathscr{O}(d_F)$ and hence $\bigoplus_1^{r_F}\mathscr{O}(-d_F)\to\mathscr{K}$. Define F by (5.2) and let $g\colon\mathscr{O}(E_p)\to\mathscr{O}(F)$ be the dual of the composed mapping $\mathscr{O}(F^*)\to\mathscr{K}\to\mathscr{O}(E^*_p)$. We then have the exact sequence

$$\mathscr{O}(F^*)\xrightarrow{g^*}\mathscr{O}(E^*_p)\xrightarrow{c^*_{p+1}}\mathscr{O}(E^*_{p+1})$$

of sheaves. We claim that

$$\mu := gR^X_p\wedge\Omega$$

is a global (vector-valued) Coleff-Herrera current. In fact, in view of (5.1),

$$\bar\partial\mu=\bar\partial gR^X_p\wedge\Omega=g\bar\partial R^X_p\wedge\Omega=gc_{p+1}R^X_{p+1}\wedge\Omega=0,$$

since $gc_{p+1}=0$. Because of the dimension principle μ must have the SEP with respect to X_{red} and hence it is, by definition, a Coleff-Herrera current and thus a section of $\mathscr{CH}^{X_{\text{red}}}\otimes F\times\mathscr{O}(N+1)$.

Let X_{p+1} be the subset of X_{red} where s_{p+1} does not have optimal rank. Let us choose a Hermitian norm on F, and define $\sigma_F\colon F\to E_p$ on the complement of Z_{p+1} so that $\sigma_F=0$ on the orthogonal complement of $\operatorname{Im} g$ and $\sigma_F g=I$ on the orthogonal complement of $\operatorname{Ker} g$. It is shown in [6, Section 2] that σ_F has an almost semi-meromorphic extension across X_{p+1}; let us denote the extension by σ_F as well. Following the proof of [27, Proposition 3.2] we see (this is just a local argument) that $R^X_p=\sigma_F gR^X_p$ outside X_{p+1}. The right hand side here is defined in view of Theorem 2.6. Since both sides have the SEP on X_{red} we conclude that they coincide in $\mathbb{P}^N$. Thus

$$R^X_p\wedge\Omega=\sigma_F\mu. \tag{5.4}$$

From [5, Theorem 4.4] we get global almost semi-meromorphic sections α_{k+1} of $\operatorname{Hom}(E_k,E_{k+1})$, $k=p,p+1,\dots$, that are smooth outside analytic subsets X_{k+1} of X_{red} where s_{k+1} do not have optimal rank, such that

$$R^X_{k+1}=\alpha_{k+1}R^X_k.$$

Since X has pure dimension it follows that $\operatorname{codim} X_{p+\ell}\ge p+\ell+1$ according to [14, Corollary 20.14]. Arguing as in the proof of [27, Proposition 3.2] we now get for each $k\ge p+1$, in view of (5.4), the representation

$$R^X_k=\alpha_k\dots\alpha_{p+1}\sigma_F\mu. \tag{5.5}$$

Now let $b_k=\alpha_k\dots\alpha_{p+1}\sigma_F$. Then b_k is an almost semi-meromorphic, see [7, Section 3.1], and by (5.5), $R^X_k=b_k\mu$ where b_k is smooth, that is, outside Z_{p+1}.

Since $\mathbf{1}_{Z_p+1}\mu = 0$ it follows from (2.5) that $R_k^X = b_k\mu$. Thus the proposition follows with $b = b_p + \cdots + b_N$.

5.2. *The current $R^a \wedge R^X$*

Assume that we have sections $a_1, \dots, a_m$ of a Hermitian line bundle S over some open set $\mathcal{U} \subset \mathbb{P}^N$ and let E be a trivial rank m bundle. Then we have interior multiplication $\delta_a \colon \Lambda^{*+1}E \otimes S^{-*-1} \to \Lambda^* E \otimes S^{-*}$, and we can consider the induced double complex as in the proof of Lemma 3.1 above. Following [8, Example 2.1] we define the Bochner-Martinelli form $U^a = U_1^a + \cdots + U_N^a$, explicitly from the a_j. The components U_k^a are almost semi-meromorphic $(0, k-1)$-forms with values in $\Lambda^k E \otimes S^{-k}$ that are smooth outside the common zero set Z_a of the a_j. Moreover, $(\delta_a - \bar\partial)U^a = 1$ outside Z_a. We thus have the residue current

$$R^a := 1 - (\delta_a - \bar\partial)U^a,$$

with support on Z_a, whose components R_k^a are $(0, k)$-currents with values in $\Lambda^k E \otimes S^{-k}$. If $\chi_\epsilon = \chi(|a|^2/\epsilon)$, where χ is a function as in (2.3) above, then $U^{a,\epsilon} = \chi_\epsilon U^a$ are smooth and tend to U^a. Thus

$$R^{a,\epsilon} = 1 - (\delta_a - \bar\partial)U^{a,\epsilon} = 1 - \chi_\epsilon + \bar\partial\chi_\epsilon \wedge U^a$$

tend to R^a. As in [8, Section 2.5], cf. (2.6) above, we can form the product

$$R^a \wedge R^X \wedge \Omega := \lim_{\epsilon \to 0} R^{a,\epsilon} \wedge R^X \wedge \Omega.$$

We will use the following important property, which follows from [8, (2.19)] and the proof [8, Lemma 2.2]:

LEMMA 5.2. *If Φ is holomorphic and $\Phi R^a \wedge R^X \wedge \Omega = 0$ at x, then Φ is in $(a)_x + \mathscr{J}_{X,x}$.*

REMARK 5.3 (Warning!). Although the components R_k^a of R^a vanish for small k because of the dimension principle, the terms $R_k^a \wedge R^X$ might be nonzero. See, e.g., [7] for examples.

5.3. *End of proof of Theorem 1.2*

To begin with we assume that $p = \operatorname{codim} Z \leq N - 1$. Let μ be the (vector-valued) Coleff-Herrera current in the representation (5.3) of $R^X \wedge \Omega$. Let us consider μ as an r_F-tuple of Coleff-Herrera currents, and let $\mathscr{L}_\alpha$, $\alpha \leq \mathfrak{m}$, be a (tuple of) Noetherian operators obtained from Theorem 4.1. Moreover, let $\widehat{M}_\alpha$ be the associated differential operators with almost semi-meromorphic coefficients so that (4.8) holds.

At a given point $x \in X_{\text{red}}$ there is a number ν_x such that if $(a) = (a_1, \dots, a_m) \subset \mathscr{O}_{X,x}$ is a local ideal, and $\phi \in \mathscr{O}_{X,x}$, then $|\mathscr{L}_\alpha \phi| \le C|a|^\nu$ on $X_{\text{red},x}$ for all $\alpha \le \mathfrak{m}$ implies that $\phi R^a \wedge R^X \wedge \Omega = 0$. This is precisely the main step of the proof of [27, Theorem 1.2] and we do not repeat it here (just notice that our number ν_x is called N in [27], our $\widetilde{M}_\alpha$ are called $\tilde{K}_\alpha$, moreover, the non-reduced space that we call X is denoted by Z in [27] whereas X denotes the associated reduced space!). In this proof the number ν_x is explicitly deduced from the singularities of the the coefficients of $\widehat{M}_\alpha$ and of b, expressed as the degree of monomials in a suitable log resolution of X_{red}, see [27, Eq. (4.9)]. In particular, the number ν_x works for all points in a neighborhood of x. By compactness we therefore get:

PROPOSITION 5.4. *There is a number ν, such that if $x \in X_{\text{red}}$, $(a) = (a_1, \dots, a_m) \subset \mathscr{O}_{X,x}$ is a local ideal, and $\phi \in \mathscr{O}_{X,x}$, then $|\mathscr{L}_\alpha \phi| \le C|a|^\nu$ on $X_{\text{red},x}$ for all $\alpha \le \mathfrak{m}$ implies that $\phi R^a \wedge R^X \wedge \Omega = 0$.*

Combined with Lemma 5.2 we have thus obtained ν and differential operators $\mathscr{L}_\alpha$ so that part (i) of Theorem 1.2 holds.

Now let F_j be polynomials as in Theorem 1.2 (ii), let f_j be the d-homogenizations considered as section of $\mathscr{O}(d)$ over X_{red} and let $\mathscr{J}_f$ be the associated ideal sheaf as in the introduction.

LEMMA 5.5. *Let Φ be a polynomial such that* (1.4) *holds and let ϕ be the ρ-homogenization of Φ. If*

$$\rho \ge \deg \Phi + \nu d^{c_\infty} \deg X_{\text{red}}, \tag{5.6}$$

then $|\mathscr{L}_\alpha \phi| \le C|f|^\nu$ for all α.

PROOF. Let $\pi\colon \tilde{X} \to X_{\text{red}}$ be the normalization of the blow-up of X_{red} along $\mathscr{J}_f$ and let $\sum r_j W_j$ be the exceptional divisor, where W_j are the irreducible components and r_j the corresponding multiplicities. Notice that if ψ is a holomorphic section of some $\mathscr{O}(\ell)$, then $|\psi| \le C|f|^\nu$ if and only if $\pi^*\psi$ vanishes to order at least νr_j on W_j for each j.

If (1.4) holds on V_{red}, then $\pi^*(\mathscr{L}_\alpha \phi)$ vanishes to order νr_j on each W_j that is not fully contained in $\pi^{-1}(X_{\text{red},\infty})$. Notice that

$$\phi = x_0^{\rho - \deg \Phi} \varphi,$$

where φ is the $\deg \Phi$-homogenization of Φ and thus holomorphic. If W_j is contained in $\pi^{-1} X_{\text{red},\infty}$, then ϕ vanishes at least to order $\rho - \deg \Phi$ on W_j. Since $\mathscr{L}_\alpha$ does not involve the derivative, $\partial/\partial x_0\ \mathscr{L}_\alpha \phi$ also vanishes to order $\rho - \deg \Phi$ on W_j. By the geometric estimate in [13], cf. [8, Eq. (6.2)], we have that

$$r_j \le d^{\operatorname{codim} \pi(W_j)} \deg X_{\text{red}}.$$

If (5.6) holds, therefore $\pi^*(\mathscr{L}_\alpha\phi)$ vanishes, at least, to order νr_j on W_j for all j. Thus the lemma follows.

With the same hypotheses as in Lemma 5.5 it follows from the lemma and Proposition 5.4 that

$$\phi R^f \wedge R^X \wedge \Omega = 0.$$

If in addition

$$\rho \geq (d-1)\min(m, n+1) + \operatorname{reg} X,$$

we can now solve a sequence of global $\bar\partial$-equations in $\mathbb{P}^N$ and get a global solution q_j to $\phi = f_1q_1 + \cdots + f_mq_m$, cf. [8, Lemma 4.3]. The fact that X is not reduced plays no role here. After dehomogenization we obtain the desired representation of Φ, and so the proof of Theorem 1.2 is complete in the case $p \leq N-1$.

Now assume that $p = \operatorname{codim} Z = N$ so that X_{red} is a finite set in $\mathbb{C}^N \simeq \mathbb{P}^N \setminus \{x_0 = 0\}$. If necessary we multiply μ by a suitable power of x_0 to be able to apply Theorem 4.1. We then get the global, in $\mathbb{C}^N$, L_α that form a complete set of Noetherian operators at each point $x \in X_{\mathrm{red}}$. Part (ii) is trivial, since the image of any ideal $(a) \subset \mathscr{O}_{X,x}$ in $\mathscr{O}_{X_{\mathrm{red}},x}$ is just either (0) or $(1) = \mathscr{O}_{X_{\mathrm{red}},x}$.

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