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ON THE x-COORDINATES OF PELL EQUATIONS WHICH ARE FIBONACCI NUMBERS

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Abstract

For an integer $d > 2$ which is not a square, we show that there is at most one value of the positive integer x participating in the Pell equation $x^2 - dy^2 = \pm 1$ which is a Fibonacci number.

1. Introduction

Let $d > 1$ be a positive integer which is not a perfect square. It is well-known that the Pell equation

$$x^2 - dy^2 = \pm 1$$

has infinitely many positive integer solutions (x, y). Furthermore, putting (x_1, y_1) for the smallest solution with $x \geq 1$, all solutions are of the form (x_n, y_n) for some positive integer n where

$$x_n + y_n\sqrt{d} = \big(x_1 + y_1\sqrt{d}\big)^n.$$

There are many papers in the literature which solve Diophantine equations involving members of the sequences $\{x_n\}_{n\geq 1}$ or $\{y_n\}_{n\geq 1}$ being squares, or perfect powers of larger exponents of some other integers, etc. (see, for example, [4], [5] and [9]). In this paper, we study a new problem of this type which we now describe.

Let $\{F_m\}_{m\geq 0}$ be the Fibonacci sequence given by $F_{m+2} = F_{m+1} + F_m$, for $m \geq 0$, where $F_0 = 0$ and $F_1 = 1$. The first terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \\ 377, 610, 987, 1597, 2584, 4181, 6765, \ldots$$

The Fibonacci numbers are well-known and possess numerous properties (see [15, pp. 53–56] and [7] together with their very extensive annotated bibliography for additional references and history).

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In this paper, we study when can x_n be a Fibonacci number, which reduces to the Diophantine equation

$$x_n \in \{F_m\}_{m\geq 1}.$$

Of course, for every integer $x \geq 2$ and every $\varepsilon \in \{\pm 1\}$, there is a unique square-free integer $d \geq 2$ such that there is an integer solution $y \in \mathbb{Z}$ to

$$x^2 - dy^2 = \varepsilon.$$

Namely d is the product of all prime factors of $x^2 - \varepsilon$ which appear at odd exponents in its factorization. In particular, taking $x = F_m$, we get that every Fibonacci number is the x-coordinate of the Pell equation corresponding to one or two specific square-free integers d. Here, we study the square-free integers d such that the sequence $\{x_n\}_{n\geq 1}$ contains at least two Fibonacci numbers. Our result is the following.

THEOREM 1.1. *Let $d \geq 2$ be square-free. The Diophantine equation*

$$x_n \in \{F_m\}_{m\geq 1} \tag{1}$$

has at most one solution (n, m) in positive integers except for $d = 2$. In this case, we have

$$(n, m) \in \{(1, 1), (1, 2), (2, 4)\}.$$

A fun reformulation of the above result is the following. Consider the Diophantine equation

$$(F_n^2 \pm 1)(F_m^2 \pm 1) = x^2 \tag{2}$$

in integers (n, m, x) with n, m positive and $x \geq 0$. To fix ideas, we assume that $n \leq m$. The above equation obviously has the trivial solutions $n = m$ and the corresponding signs being equal, as well as $x = 0$, $n \in \{1, 2\}$, and its corresponding sign being negative. Theorem 1.1 implies that the only nontrivial solutions are $(n, m, x) = (1, 3, 4), (2, 3, 4)$. Indeed this deduction relies on the fact that if a product of two positive integers is a square, then each one of them is of the form d times a perfect square for the same square-free integer d. Assuming $x > 0$ and applying the above argument to the left-hand side of equation (2) we arrive at the problem treated by Theorem 1.1.

The organization of this paper is as follows. The proof of Theorem 1.1 proceeds in two cases according to whether n is even or odd. So in Section 2, we consider n even and prove that equation (1) has more that one solution if and only if $d = 2$. In this case, the solutions are those listed in Theorem 1.1. In fact, we transform the main problem into finding integer points of some elliptic curves. This is done by the means of MAGMA. In Section 3, we take n odd and use Baker's method and the Baker-Davenport reduction method to prove that there is no other solution than those obtained in the case of n even.

2. The case n even

Let

$$\alpha = x_1 + y_1\sqrt{d}, \quad \beta = x_1 - y_1\sqrt{d} = \varepsilon\alpha^{-1}, \qquad \varepsilon \in \{\pm 1\}.$$

Then,

$$x_n + y_n\sqrt{d} = \alpha^n,$$

which leads to

$$x_n = \frac{\alpha^n + \beta^n}{2}.$$

Write $n = 2n_0$. Since

$$x_n = x_{2n_0} = x_{n_0}^2 + dy_{n_0}^2 = 2x_{n_0}^2 \pm 1,$$

it suffices to solve the equation

$$2u^2 \pm 1 = F_m, \qquad \text{where} \quad m \geq 1. \tag{3}$$

There are many papers in the literature solving Diophantine equations of the form $F_n = f(u)$, for some quadratic polynomial $f(x) \in \mathbb{Q}[x]$ by elementary means. We give only a couple of examples. The only squares in the Fibonacci sequence are $0 = F_0$, $1 = F_1 = F_2$, $144 = F_{12}$. This is a consequence of the work of Ljunggren [8], [10] (see the Introduction to [11]) and was rediscovered by Cohn [3] and Wyler [16]. All triangular numbers in the Fibonacci sequence are $1 = F_1 = F_2$, $3 = F_4$, $21 = F_8$, $55 = F_{10}$ were found by an elementary method by Luo Ming [14]. It is therefore likely that one can find all solutions of equation (3) by elementary means using only congruences and Jacobi symbol manipulations. We preferred a more computational approach using MAGMA, which we now describe. Since the formula

$$V^2 - 5U^2 = \pm 4, \tag{4}$$

holds with $(V, U) = (L_m, F_m)$, where $\{L_n\}_{n\geq 0}$ is the Lucas companion of the Fibonacci sequence given by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, for all $n \geq 0$, it follows that by replacing F_m with $2u^2 \pm 1$ and setting $v = L_m$, we obtain

$$v^2 = 5(2u^2 \pm 1)^2 \pm 4. \tag{5}$$

In the right-hand sides of (5) above we have four polynomials, each of degree 4. Se we are lead to integer points (u, v) on the following four elliptic curves:

$$v^2 = 20u^4 + 20u^2 + 9; \tag{6}$$
$$v^2 = 20u^4 + 20u^2 + 1; \tag{7}$$
$$v^2 = 20u^4 - 20u^2 + 9; \tag{8}$$
$$v^2 = 20u^4 - 20u^2 + 1. \tag{9}$$

We used MAGMA to determine the integer points (u, v) on these elliptic curves. We obtained:

$$\begin{aligned} (0, \pm 3), (\pm 1, \pm 7), &\quad \text{for curve (6);} \\ (0, \pm 1), &\quad \text{for curve (7);} \\ (0, \pm 3), (\pm 1, \pm 3), &\quad \text{for curve (8);} \\ (0, \pm 1), (\pm 1, \pm 1), &\quad \text{for curve (9).} \end{aligned}$$

As $F_m = 2u^2 \pm 1$, we get that $x_n = F_m \in \{1, 3\}$. Since n is even, the only possibility is $d = 2$, $n = 2$, $m = 4$. Thus,

$$x^2 - 2y^2 = \pm 1$$

has $x_2 = F_4 = 3$. Actually, we solve completely the problem in the case $d = 2$ in the following lemma.

LEMMA 2.1. *Assume that $x^2 - dy^2 = \pm 1$ and that $x_n = F_m$, for some even n. Then $d = 2$. Further, all solutions (n, m) (regardless of the parity of n) are*

$$(n, m) \in \{(1, 1), (1, 2), (2, 4)\}.$$

PROOF. Obviously, $x_1 = 1 = F_1 = F_2$. Assume for a contradiction that $x_n = F_m$ and $n > 2$. From the previous arguments we know that n is odd. Assume $m \neq 1, 2, 4$. Let x be the common value of $x_n = F_m$. Since n is odd, we have $x^2 + 1 = 2y^2$, where $y = y_n$. On the other hand, by (4), we also have $5x^2 \pm 4 = v^2$, where $v = L_m$. Multiplying these two relations, we get

$$(x^2 + 1)(5x^2 \pm 4) = 2z^2,$$

where $z := yv$. We get the two equations

$$(2z)^2 = 10x^4 + 18x^2 + 8;$$
$$(2z)^2 = 10x^4 + 2x^2 - 8.$$

With MAGMA we got $(x, 2z) = (\pm 1, \pm 2), (\pm 1, \pm 6)$. That is,

$$(x, z) = (\pm 1, \pm 1), (\pm 1, \pm 3),$$

none of which leads to a new convenient solution to our original problem.

3. The case n odd

3.1. Preliminary considerations

From now on, $d > 2$. Thus, we may assume that $n_2 = n_1 n$ and $m_2 = m_1 t$ with n odd and $t \in \mathbb{Z}$. We denote in this whole section x_{n_1} by x_1 and y_{n_1} by y_1, then $\alpha = x_{n_1} + y_{n_1}\sqrt{d}$ and $\beta = x_{n_1} - y_{n_1}\sqrt{d}$ so that

$$x_{n_1} = F_{m_1}, \qquad x_{n_2} = F_{m_2}.$$

But $\gcd(x_{n_1}, x_{n_2}) = \gcd(F_{m_1}, F_{m_2})$ implies that $x_{\gcd(n_1,n_2)} = F_{\gcd(m_1,m_2)}$. Indeed, the fact that $\gcd(x_{n_1}, x_{n_2}) = x_{\gcd(n_1,n_2)}$ when $n_1,\ n_2$ are odd follows from (ii) of Theorem 0 in [13]. The fact that the same is true for Fibonacci numbers even without the restriction that indices are odd is (i) of the same theorem. So by performing the replacement $(n_1, m_1) \to ((n_1, n_2), (m_1, m_2))$, we may assume that $n_1 \mid n_2$ and $m_1 \mid m_2$. We change $x_{n_1} \to x_1$ and $(dy_{n_1}^2 \to dy_1^2)$. So, we have

$$x_1 = \frac{\alpha + \beta}{2} = F_{m_1} \tag{10}$$

and

$$x_n = \frac{\alpha^n + \beta^n}{2} = F_{m_1 t}, \tag{11}$$

where we have put $t := m_2/m_1$. We also put $\varepsilon = \alpha\beta \in \{\pm 1\}$. With these notations, the following inequalities hold.

LEMMA 3.1. *We have the following estimates:*

$$\left|\alpha - \frac{2}{\sqrt{5}}\gamma^{m_1}\right| < \frac{4}{\gamma^{m_1}}; \tag{12}$$

$$\gamma^{m_1 t - 2} < \alpha^n < \gamma^{m_1 t + 2}; \tag{13}$$

$$\left|\frac{\sqrt{5}}{2}\gamma^{-m_1 t}\alpha^n - 1\right| < \frac{2\sqrt{5}}{\gamma^{2m_1 t}}. \tag{14}$$

PROOF. Using equation (10) and the Binet formula for the Fibonacci numbers, we have

$$\frac{\alpha + \beta}{2} = \frac{\gamma^{m_1} - \delta^{m_1}}{\sqrt{5}},$$

where $\gamma = (1+\sqrt{5})/2$ and $\delta = (1-\sqrt{5})/2$. We deduce that

$$\alpha = \frac{2}{\sqrt{5}}\gamma^{m_1} - \beta - \frac{2}{\sqrt{5}}\delta^{m_1}. \tag{15}$$

Since $\alpha > 3$ (because $d > 2$) and $|\beta| < 1$, we have

$$\frac{\alpha}{3} < \frac{\alpha+\beta}{2} < \alpha.$$

Further, we see that

$$\gamma^{m_1-2} < F_{m_1} < \gamma^{m_1-1}.$$

Thus, from (10), we deduce

$$\frac{\alpha}{3} < \gamma^{m_1-1} \qquad \text{so} \quad \alpha < 3\gamma^{m_1-1} < \gamma^{m_1+2},$$

as well as

$$\gamma^{m_1-2} < F_{m_1} < \alpha.$$

So, we get

$$\gamma^{m_1-2} < \alpha < \gamma^{m_1+2}.$$

Therefore, from (15), we have

$$\left|\alpha - \frac{2}{\sqrt{5}}\gamma^{m_1}\right| = \left|\frac{\pm 1}{\alpha} + \frac{2}{\sqrt{5}}(\pm\gamma)^{m_1}\right| \le \frac{1}{\gamma^{m_1}}\left(\frac{2}{\sqrt{5}} + \gamma^2\right) < \frac{4}{\gamma^{m_1}},$$

which leads to (12). On the other hand, we use equation (11) to get

$$\alpha^n = \frac{2}{\sqrt{5}}\gamma^{m_1 t} - \beta^n - \frac{2}{\sqrt{5}}\delta^{m_1 t}. \tag{16}$$

Similarly as above, we have

$$\begin{aligned}\gamma^{m_1 t-2} < F_{m_1 t} = \frac{\alpha^n+\beta^n}{2} &< \alpha^n \\ &< 3\left(\frac{\alpha^n+\beta^n}{2}\right) = 3F_{m_1 t} < 3\gamma^{m_1 t-1} < \gamma^{m_1 t+2}.\end{aligned}$$

The above inequality now implies (13). Further, estimate (13) together with (16) leads to

$$\left|\alpha^n - \frac{2}{\sqrt{5}}\gamma^{m_1 t}\right| = \left|\frac{\pm 1}{\alpha^n} + \frac{2}{\sqrt{5}}(\pm\gamma)^{m_1 t}\right| \le \frac{1}{\gamma^{m_1 t}}\left(\frac{2}{\sqrt{5}} + \gamma^2\right) < \frac{4}{\gamma^{m_1 t}};$$

which easily leads to (14). This completes the proof of Lemma 3.1.

3.2. The case $m_1 = 3$

In this case, $x_1 = F_{m_1} = 2$. Thus, $d \in \{3, 5\}$. If $d = 5$, we then have $x_1^2 - 5y_1^2 = -1$. Since n is odd, we have that $x_n^2 - 5y_n^2 = -1$. Hence, $(2x_n)^2 - 5(2y_n)^2 = -4$. It is known that all integer solutions (U, V) of the equation (4) are of the form $(U, V) = (F_k, L_k)$, for some integer k and $L_k^2 - 5F_k^2 = 4(-1)^k$. For us, we get $2F_{m_1t} = L_k$ with k odd, so $F_{m_1t} = L_k/2$. Thus, $2F_{m_1t} = L_k = F_{2k}/F_k$, so $F_{2k} = 2F_{m_1t}F_k$. If $k \geq 7$, then, by Carmichael's Theorem on Primitive Divisors (see [2]), F_{2k} has a primitive divisor which cannot divide $2F_{m_t}F_k$. Thus, $k \leq 6$ and since F_{2k} is even, it follows that $3 \mid k$. Thus, $k \in \{3, 6\}$. The situation $k = 3$ leads to $8 = F_6 = 2F_{m_1t}F_3 = 4F_{m_1t}$, so $F_{m_1t} = 2$, therefore $t = 1$, which is not convenient, while the situation $k = 6$ leads to $144 = F_{12} = 2F_{m_1t}F_6 = 16F_{m_1t}$, so $F_{m_1t} = 9$, which is impossible.

For the case $d = 3$, we appeal again to MAGMA. Namely, in this case with $(x, y) = (x_n, y_n)$, we have $x^2 - 3y^2 = 1$, so $x^2 - 1 = 3y^2$. Since by (4) we also have $5x^2 \pm 4 = v^2$, where $v = L_{m_1t}$, we get that

$$(x^2 - 1)(5x^2 \pm 4) = 3z^2,$$

where $z := yv$. Expanding, we get

$$(3z)^2 = 15x^4 - 3x^2 - 12;$$
$$(3z)^2 = 15x^4 - 27x^2 + 12.$$

With MAGMA we got $(x, 3z) = (\pm 1, 0), (\pm 2, \pm 12)$, i.e. $(x, z) = (\pm 1, 0)$, $(\pm 2, \pm 6)$. Therefore, we have no new solution.

From now on, we assume that $m_1 \geq 4$.

3.3. An inequality among n and t

In this subsection, we prove the following result that helps to compare n and t.

LEMMA 3.2. *If equation* (11) *has a solution, then we have $n > t$.*

PROOF. Note that

$$(\alpha, \beta) = \left(F_{m_1} + \sqrt{F_{m_1}^2 - \varepsilon},\, F_{m_1} - \sqrt{F_{m_1}^2 - \varepsilon}\right).$$

By induction, it is readily proved that the two sequences $\{x_n\}_{n\geq 1}$ and $\{F_{m_1n}\}_{n\geq 1}$ satisfy

$$x_n = 2F_{m_1}x_{n-1} + (-\varepsilon)x_{n-2}; \tag{17}$$

$$F_{m_1n} = L_{m_1}F_{m_1(n-1)} + (-1)^{m_1-1}F_{m_1(n-2)} \tag{18}$$

for all $n \geq 3$. Further,

$$x_1 = F_{m_1}, \qquad x_2 = 2F_{m_1}^2 - \varepsilon \leq 2F_{m_1}^2 + 1 < F_{2m_1}. \tag{19}$$

The last inequality above follows because $F_{2m_1} = F_{m_1}L_{m_1}$ and $L_{m_1} > 2F_{m_1}$, for $m_1 \geq 4$, inequality which is obvious in light of the formula $L_{m_1} = 2F_{m_1} + F_{m_1-3}$, which can be proved by induction on $m_1 \geq 4$. We now prove by induction on n that the inequality

$$x_n < F_{m_1 n} \quad \text{holds,} \qquad \text{for all} \quad n \geq 2.$$

This together with the fact that for us $x_n = F_{m_1 t}$, will give us the desired conclusion that $t < n$.

The inequality $x_n < F_{m_1 n}$ holds with $n = 2$ by (19) and we also have $x_1 = F_{m_1}$ (so when $n = 1$ we have equality). Suppose that $n \geq 3$. Since $L_{m_1} > 2F_{m_1}$, for all $m_1 \geq 4$, the desired inequality follows by induction on n from the two recurrences (17) and (23) when m_1 is odd. When m_1 is even, we have, again by induction on n,

$$\begin{aligned} F_{m_1 n} &= L_{m_1} F_{m_1(n-1)} - F_{m_1(n-2)} \\ &= (L_{m_1} - 1)F_{m_1(n-1)} + (F_{m_1(n-1)} - F_{m_1(n-2)}) \\ &\geq 2F_{m_1} F_{m_1(n-1)} + F_{m_1(n-2)} > 2F_{m_1} x_{n-1} + x_{n-2} = x_n, \end{aligned}$$

which is what we wanted to prove.

3.4. An inequality among m_1 and n

The following result will help to compare m_1 and n.

LEMMA 3.3. *If equation* (11) *has a solution, then we have* $\gamma^{m_1} < 6n^2$.

PROOF. We shall show that

$$F_{m_1} \mid n^2 \pm t^2. \tag{20}$$

The right-hand side above is nonzero by Lemma 3.2. Divisibility (20) will immediately imply the desired conclusion since then $\gamma^{m_1-2} < F_{m_1} \leq n^2 \pm t^2 < 2n^2$ by Lemma 3.2, so $\gamma^{m_1} < 2\gamma^2 n^2 < 6n^2$, which is what we want.

Recall that the Dickson polynomial

$$D_n(x, v) = \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{n}{n-p} \binom{n-p}{p} (-v)^p x^{n-2p} \tag{21}$$

satisfies

$$D_n(u+v/u, v) = u^n + (v/u)^n.$$

Taking n to be odd, $u=\alpha$, $v=\varepsilon$, we get that

$$\frac{x_n}{x_1} = \frac{\alpha^n+\beta^n}{\alpha+\beta} = \frac{D_n(2x_1,\varepsilon)}{2x_1} \equiv (-\varepsilon)^{\lfloor n/2\rfloor} n \pmod{x_1},$$

by (21). Since $x_1 = F_{m_1}$ and $x_n = F_{m_1 t}$, we get that

$$\frac{F_{m_1 t}}{F_{m_1}} \equiv \pm n \pmod{F_{m_1}}. \tag{22}$$

When t is odd, the left-hand above is congruent to $\pm t$ modulo F_{m_1}, a fact which can be proved invoking again properties of the Dickson polynomials. But we prefer a direct approach. Given two algebraic integers η, ζ and an integer m we say that $\eta \equiv \zeta \pmod m$ if $(\eta-\zeta)/m$ is an algebraic integer. Then, $\gamma^{m_1} \equiv \delta^{m_1} \pmod{F_{m_1}}$, therefore

$$\frac{F_{m_1 t}}{F_{m_1}} = \frac{\gamma^{m_1 t}-\delta^{m_1 t}}{\gamma^{m_1}-\delta^{m_1}} = \gamma^{m_1(t-1)} + \cdots + \delta^{m_1(t-1)} \equiv t\gamma^{m_1(t-1)} \pmod{F_{m_1}}.$$

The same congruence holds if we replace γ by δ and multiplying them we get

$$\left(\frac{F_{m_1 t}}{F_{m_1}}\right)^2 \equiv t^2(\gamma\delta)^{m_1(t-1)} \equiv \pm t^2 \pmod{F_{m_1}}. \tag{23}$$

By (22), the left-hand side above is congruent to $n^2 \pmod{F_{m_1}}$, which together with (23) leads to divisibility relation (20), which is what we wanted.

3.5. Bounding n and m_1

The next result to prove will give us upper bounds for n and m_1. But before this, we recall the following result due to Matveev [12]. Let $\mathbb{L}$ be an algebraic number field and $d_{\mathbb{L}}$ be the degree of the field $\mathbb{L}$. Let $\eta_1, \eta_2, \ldots, \eta_\ell \in \mathbb{L}$ not 0 or 1 and $d_1, \ldots, d_\ell$ be nonzero integers. We put

$$D = \max\{|d_1|, \ldots, |d_\ell|, 3\},$$

and put

$$\Lambda = \prod_{i=1}^{\ell} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_\ell$ be positive numbers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log|\eta_j||, 0.16\}, \qquad \text{for} \quad j = 1, \ldots, \ell,$$

where for an algebraic number η we write $h(\eta)$ for its Weil height. The result below follows from Corollary 2.3 of Matveev [12] and its proof was worked out as Theorem 9.4 in [1].

THEOREM 3.4. *If* $\Lambda \neq 0$ *and* $\mathbb{L} \subset \mathbb{R}$, *then*

$$\log|\Lambda| > -1.4 \cdot 30^{\ell+3}\ell^{4.5} d_{\mathbb{L}}^2 (1+\log d_{\mathbb{L}})(1+\log D) A_1 A_2 \cdots A_\ell.$$

We will use the above theorem to prove the following result.

LEMMA 3.5. *We have* $n < 7 \times 10^{14}$. *Additionally, we have* $m_1 \leq 145$.

PROOF. We take

$$\Lambda := \frac{\sqrt{5}}{2}\gamma^{-m_1 t}\alpha^n - 1.$$

This is nonzero, since if it were, then $\sqrt{5}/2 = \gamma^{m_1 t}\alpha^{-n}$ would be a unit, which is false since it belongs to $\mathbb{K} = \mathbb{Q}(\sqrt{5}\,)$ and its norm from $\mathbb{K}$ to $\mathbb{Q}$ is $5/4$. We use Theorem 3.4 to get a lower bound for $|\Lambda|$. We take $\ell = 3$,

$$\eta_1 = \sqrt{5}/2, \quad \eta_2 = \gamma, \quad \eta_3 = \alpha, \qquad d_1 = 1, \quad d_2 = -m_1 t, \quad d_3 = n.$$

We put $\mathbb{L} := \mathbb{Q}(\sqrt{5}, \alpha)$. Clearly, $d_{\mathbb{L}} \in \{2, 4\}$. We have $h(\eta_1) = (\log 5)/2$, $h(\eta_2) = (\log\gamma)/2$, $h(\alpha) = (\log\alpha)/2$. Thus, we can take $A_1 = 2\log 5$, $A_2 = 2\log\gamma$, $A_3 = 2\log\alpha$. Since $d \geq 3$, we have that $\alpha \geq 2 + \sqrt{3} > \gamma^2$, so inequality (13) gives that

$$\gamma^{2n} < \alpha^n < \gamma^{m_1 t + 2},$$

therefore $2n \leq m_1 t + 1$, so $n < m_1 t$. Hence, we can take $D := m_1 t$. Theorem 3.4 gives now that

$$-\log|\Lambda| < 1.4 \times 30^6 \times 3^{4.5} \times 4^2 (1+\log 4)(2\log 5)(2\log\gamma)(2\log\alpha)(1+\log(m_1 t)). \quad (24)$$

On the other hand, inequalities (13) and (14) give

$$|\Lambda| < \frac{2\sqrt{5}}{\gamma^{2m_1 t}} < \frac{2\sqrt{5}\gamma^4}{\alpha^{2n}} < \frac{31}{\alpha^{2n}} \qquad \text{so} \quad -\log|\Lambda| > 2n\log\alpha - \log 31. \quad (25)$$

Putting (24) and (25) together, we get

$$n < 1.4 \times 30^6 \times 3^{4.5} \times 4^2 (1+\log 4)(2\log 5)(2\log\gamma)(1+\log(m_1 t)) + \frac{\log 31}{\log\alpha}.$$

Since $\alpha \geq 2+\sqrt{3}$, $t < n$ (by Lemma 3.2) and $m_1 < \log(6n^2)/\log\gamma$ (by Lemma 3.3), we get

$$n < 1.7 \times 10^{13}(1+\log(n\log(6n^2)/\log\gamma)),$$

giving $n < 7 \times 10^{14}$. Additionally, $F_{m_1} < 2n^2 < 10^{30}$ (see the proof of Lemma 3.3), so $m_1 \leq 145$.

3.6. The final step

For each $m_1 \in [4, 145]$ and $\varepsilon \in \{\pm 1\}$, we calculate

$$\alpha = F_{m_1} + \sqrt{F_{m_1}^2 - \varepsilon}.$$

We put

$$\Gamma := n\log\alpha - m_1 t\log\gamma + \log(\sqrt{5}/2).$$

Note that $e^\Gamma - 1 = \Lambda$. Since $t \geq 2$, $m_1 \geq 4$, we have that $m_1 t \geq 8$, so by (14), we have that

$$|\Lambda| < \frac{2\sqrt{5}}{\gamma^{2m_1 t}} < \frac{1}{2}.$$

By a classical inequality, this leads to

$$|\Gamma| \leq 2|\Lambda| \leq \frac{4\sqrt{5}}{\gamma^{2m_1 t}}. \tag{26}$$

Inequality (26) is suitable to apply the reduction algorithm. Note that $n < m_1 t < m_1 n < 1.2 \times 10^{17} := M$. So in order to deal with the remaining cases, for $m_1 \in [4, 145]$, we used a Diophantine approximation algorithm called the Baker-Davenport reduction method. The following lemma is a slight modification of the original version of Baker-Davenport reduction method. (See [6, Lemma 5a]).

LEMMA 3.6. *Let κ and μ be given real numbers. Assume that M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < m\kappa - n + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log(AQ/\eta)}{\log B} \le m \le M.$$

As

$$0 < n\log\alpha - m_1 t\log\gamma + \log(\sqrt{5}/2) < \frac{4\sqrt{5}}{\gamma^{2n}},$$

we apply Lemma 3.6 with

$$\kappa = \frac{\log\alpha}{\log\gamma}, \quad \mu = \frac{\log(\sqrt{5}/2)}{\log\gamma}, \quad A = \frac{4\sqrt{5}}{\log\gamma}, \quad B = \gamma^2, \quad M = 1.2\cdot 10^{17}.$$

The program was developed in PARI/GP running with 200 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the conditions. In one minute all the computations were done. In all cases, we obtained $m_1 t \le 151$. We set $M = 151$ to check again in the range $4 \le n \le 151$. The second run of the reduction method yields $m_1 t \le 149$ and then $n \le 149$. For the third round, we consider the following ranges and obtained better bounds:

(1) $4 \le n \le 50$, then $M = 54$;
(2) $51 \le n \le 100$, then $M = 105$;
(3) $101 \le n \le 145$, then $M = 149$.

For each t, we choose n odd such that inequalities (13) holds (if it exists) and with this n, we check whether the equality

$$2x_n = D_n(F_{m_1}, \varepsilon) = 2F_{m_1 t},$$

holds where the polynomial $D(x, v)$ is shown at (21). It does if and only if we have found another solution to our original problem. We wrote a program in Maple that we ran and we found no new solutions.

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