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TOPOLOGICAL RIGIDITY OF QUASITORIC MANIFOLDS

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Abstract

Quasitoric manifolds are manifolds that admit an action of the torus that is locally the same as the standard action of T^n on $\mathbb{C}^n$. It is known that the quotients of such actions are nice manifolds with corners. We prove that a class of locally standard manifolds, that contains the quasitoric manifolds, is equivariantly rigid, i.e., that any manifold that is T^n-homotopy equivalent to a quasitoric manifold is T^n-homeomorphic to it.

1. Introduction

Toric varieties are studied extensively in algebraic geometry and combinatorics ([6], [13]). The main tool in their study is the polytope that is determined by the fan of the toric variety. Actually, this polytope is the quotient of the torus action on the toric variety. The combinatorial properties of the polytope reflect the algebraic and geometric properties of the variety and vice versa. A topological analogue of toric varieties was introduced by Davis-Januszkiewicz [4] and called toric manifolds in their paper. To avoid confusion with the terminology, later the term *quasitoric manifolds* became prominent for these spaces. The term "toric manifold" is reserved for non-singular toric varieties. Quasitoric manifolds are manifolds that admit a *locally standard* action of the torus T^n such that the quotient space is a simple polytope. More precisely, a T^n-action on a manifold M^{2n} is locally standard if T^n acts locally by the standard coordinate-wise multiplication on $\mathbb{C}^n$. As in the toric variety case, the combinatorial properties of the polytope provide information about the topological structure of the quasitoric manifold. Furthermore, the manifolds can be reconstructed from the polytope and an appropriate assignment of subgroups of T^n to its faces.

In the present work, we study general locally standard T^n-actions on manifolds. In this case, the quotient space is just a nice manifold with corners. As

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before, the combinatorial properties of the manifold with corners are reflected to the topology of the original manifold. Also, the manifold itself can be reconstructed by an appropriate assignment of subgroups of T^n to the faces of the manifold with corners.

Let L be a Lie group. An L-manifold X is called *locally linear* if for every point $x \in X$, there is a linear L_x-slice, i.e., a slice that is equivalent to an orthogonal L_x-representation [1, p. 171]. Now we can state the main theorem.

MAIN THEOREM. *Let M^{2n} be a closed T^n-quasitoric manifold, N^{2n} a closed locally linear T^n-manifold and $f\colon N^{2n} \to M^{2n}$ an equivariant homotopy equivalence. Then f is equivariantly homotopic to an equivariant homeomorphism.*

Actually, the theorem is proved for a slightly more general class of locally standard torus manifolds (Theorem 4.11).

The idea of the proof is the same as the one used in the Coxeter group case ([12], [17], [18]). After all, the reconstruction of the quasitoric and locally standard torus manifolds from their quotient spaces is similar to the construction of the Coxeter manifold from the Coxeter complex of a Coxeter group. That similarity was made precise in [4]. First we show that N^{2n} is a locally standard torus manifold. Let $X = M^{2n}/T^n$ and $Y = N^{2n}/T^n$. Then X and Y are nice manifolds with corners and f induces a map $\phi\colon Y \to X$ that is a face-preserving homotopy equivalence. As in the references for the Coxeter group case, we show inductively that there is a face-preserving homotopy from ϕ to a face-preserving homeomorphism h. The homeomorphism h lifts to a T^n-homeomorphism between N^{2n} and M^{2n} that is T^n-homotopic to f. It should be noticed that even though the result is in the spirit of equivariant surgery theory, the method of proof uses little from surgery methods. It is based mainly on the combinatorics of the quotient space and the nice local properties of the quotient map, as in the Coxeter group case.

The main theorem, loosely, can be considered as a version of an equivariant or stratified Borel Conjecture. Let $\pi\colon M^{2n} \to X$ be the quotient map. Over the interior $\mathring{\sigma}$ of some face σ of X, the map π is a fiber bundle with fiber T_σ, where T_σ is the isotropy group of σ. So M^{2n} admits a stratification by open aspherical manifolds. The proof that N^{2n} is locally standard actually shows that f is a stratified homotopy equivalence.

There are rigidity results known for non-singular toric varieties ([9], [11]), for quasitoric manifolds ([21], [22]) and for locally standard torus manifolds [24]. In all the above, a comparison is done between two quasitoric manifolds and the classification is given using cohomological and combinatorial data associated to the spaces.

In [24], a generalization of locally standard actions is given, called local torus actions. Our methods do not directly generalize to this case. In [23], a generalization of the quotient map $\pi\colon M^{2n} \to X$ is given. It is called a locally standard torus fibration. Again, our methods can not be applied directly to the stratified rigidity problem for such M^{2n}.

Finally, the torus manifolds defined in [10] generalize the quasitoric manifolds. Since the orbit spaces of the action, in general, do not have the combinatorial structure of the ones of the quasitoric manifolds, our methods do not apply directly.

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2. Preliminaries and notation

We consider S^1 as the standard subgroup of $\mathbb{C}^*$, namely the multiplicative group of non-zero complex numbers. Similarly, we consider the torus T^n as a subgroup of $(\mathbb{C}^*)^n$. We refer to the standard representation of T^n by diagonal matrices in $U(n)$ as the standard action of T^n on $\mathbb{C}^n$. The orbit space of the action is the positive cone $\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n) : x_i \geq 0\}$.

DEFINITION 2.1. Let M^{2n} be a $2n$-dimensional manifold with an action of T^n. Let M^{T^n} denote the fixed point set of M^{2n} under the T^n action. The action is called *locally standard* if:

(1) it is effective,

(2) $M^{T^n} \neq \emptyset$ i.e., there is a T^n-fixed point,

(3) for every $x \in M^{2n}$ there is a T^n invariant neighborhood U of x, a homeomorphism $f\colon U \to W$ where W is an open set in $\mathbb{C}^n$ invariant under the standard action of T^n, and an automorphism $\phi\colon T^n \to T^n$ such that $f(ty) = \phi(t)f(y)$ for all $y \in U$.

A $2n$-dimensional manifold M^{2n} with a locally standard action of T^n is called a *locally standard torus manifold*. We will consider only closed locally standard torus manifolds.

REMARK 2.2.

(1) In [10], a generalization of locally standard torus manifolds is defined. They are called torus manifolds and they are smooth T^n-manifolds with an effective action and with non-trivial T^n-fixed point set. In general, they do not satisfy the third part of the above definition.

(2) If M^{2n} is a smooth torus manifold (as in (1) above) and $H^{\mathrm{odd}}(M^{2n}) = 0$, then the T^n action is locally standard [10].

The next definition formalizes the local properties of the quotient space of a locally standard T^n action.

DEFINITION 2.3. A space X^n is an *n-manifold with corners* if it is a Hausdorff, second countable space equipped with an atlas of open sets homeomorphic to open subsets of $\mathbb{R}^n_+$ such that the overlap maps are local homeomorphisms that preserve the natural stratification of $\mathbb{R}^n_+$ [3].

Let X be an n-manifold with corners. For each $x \in X$ and each chart σ of x, define $c(x)$ to be the number of coordinates of $\sigma(x)$ that are 0. The number $c(x)$ is independent of the choice of the chart σ and so c defines a map $c\colon X \to \mathbb{N}$. For $0 \leq k \leq n$, a connected component of $c^{-1}(k)$ is called a *preface* of codimension k. The closure of a preface of codimension k is called a codimension-k *face* or an $(n-k)$-dimensional face. The manifold X itself is a codimension-0 face. The codimension-1 faces are called *facets*, the codimension-n faces are called *vertices* and the codimension-$(n-1)$ faces are called *edges*. A manifold with corners X is called *nice* if

(1) for every $0 \leq k \leq n$ there is a codimension-k face,
(2) for each codimension-k face F, there are exactly k facets $F_1, \dots, F_k$ such that F is a connected component of $F_1 \cap \cdots \cap F_k$. Moreover F does not intersect any other facet.

A nice manifold with corners X is a *homotopy polytope* if all prefaces are contractible (in particular they are connected). The k-*skeleton* of a manifold with corners X is the set of all faces of dimension less than or equal to k and it is denoted by $X^{(k)}$.

The following remark summarizes the connection between locally standard torus manifolds and manifolds with corners.

REMARK 2.4.
(1) Let M^{2n} be a closed locally standard torus manifold. Then the quotient space $X = M^{2n}/T^n$ is a compact nice n-manifold with corners ([8], [10], [11], [24]).
(2) As we mentioned already, quasitoric manifolds are locally standard torus manifolds with the property that the quotient space is not just a manifold with corners; it is a simple polytope.
(3) Let M^{2n} be a locally standard torus manifold with $\pi\colon M^{2n} \to X$ the orbit map. Then points in M^{2n}, with the same isotropy groups, are mapped to the relative interior of a preface of X. Thus the action of T^n is free over the open stratum of X and the vertices of X, i.e., the 0-dimensional faces, correspond to the global fixed points of the action.

DEFINITION 2.5. Let M^{2n} be a locally standard torus manifold with $X = M^{2n}/T^n$ the quotient manifold with corners and $\pi\colon M^{2n} \to X$ the quotient map. Then M^{2n} is called a T^n-manifold over X.

Let $\pi\colon M^{2n} \to X$ be the projection defined above. A codimension-1 connected component of a fixed point set of a circle in T^n is called a *characteristic submanifold* of M^{2n} [2, p. 34]. The images of the characteristic submanifolds are the facets of X.

3. The canonical model

In the present section we will show how to construct a locally standard torus manifold from an n-manifold with corners X and some linear data on the set of facets of X. We use the construction in [10] that generalizes the construction of quasitoric manifolds in [2] and [4]. For simplicity we set $T = T^n$.

First, we will see some of the properties of characteristic submanifolds of a locally standard torus manifold. We assume that M^{2n} is a closed locally standard torus manifold and thus the quotient $X = M^{2n}/T$ is a nice closed n-manifold with corners. For each facet X_i of X, let $M_i^{2(n-1)} = \pi^{-1}(X_i)$ be the corresponding characteristic submanifold ($i = 1, \ldots, k$). Let

$$\Lambda\colon \{X_1, \ldots, X_k\} \to \{T' \mid T' < T,\ T' \text{ 1-dimensional}\}$$

be the map that assigns to each X_i the isotropy group of the corresponding characteristic manifold $M_i^{2(n-1)}$. More precisely, $\Lambda(X_i)$ has the form

$$T_{X_i} = \{(e^{2\pi i \lambda_{1j}\phi}, \ldots, e^{2\pi i \lambda_{nj}\phi}) \in T^n : \phi \in \mathbb{R}\},$$

for some primitive vector $(\lambda_1, \ldots, \lambda_n)$ of $\mathbb{Z}^n$. The main property of these data is the following (see [2, p. 34]):

PROPERTY (*): if $X_{i_1} \cap \cdots \cap X_{i_m} \neq \emptyset$ then the induced map $\Lambda(X_{i_1}) \times \cdots \times \Lambda(X_{i_m}) \to T$ is injective.

From now on, we will identify $\Lambda(X_{i_1}) \times \cdots \times \Lambda(X_{i_m})$ with its image in T.

Let F be a k-face of X. Then F is a component of $X_{i_1} \cap \cdots \cap X_{i_{n-k}}$, for some facets X_{i_j} of X. Let $T_F = T_{X_{i_1}} \times \cdots \times T_{X_{i_{n-k}}}$, which is an $(n-k)$-torus. That construction defines a map between lattices, extending the map Λ above [2, p. 34].

$$\Lambda\colon \{F \mid F < X\} \to \{T' \mid T' < T\}, \quad F \mapsto T_F.$$

Now, we give the inverse of the above construction [2, Construction 2.2.2]. Start with a compact manifold with corners X and a map Λ that satisfies Property (*) above. Such a pair (X, Λ) is called a *characteristic pair* and Λ a *characteristic map*. For $x \in X$, we denote by $F(x)$ the smallest face of X that contains x in its relative interior. Define:

$$M_X(\Lambda) = T \times X/\sim, \quad (t, x) \sim (t', x') \iff x = x' \text{ and } t^{-1}t' \in T_{F(x)}.$$

The space $M_X(\Lambda)$ is a closed manifold and the torus T acts on it by acting on the first coordinate. In fact, the space $M_X(\Lambda)$ is a locally standard torus manifold ([2, Construction 2.2.2], [10, Lemma 4.5]).

The following result is implicit in [2, p. 34]. We give here a proof for completeness.

LEMMA 3.1. *Let (X, Λ) be a characteristic pair with $M_X(\Lambda)$ the corresponding canonical locally standard torus manifold. Let $\pi\colon M_X(\Lambda) \to X$ be the quotient map. Then for a face F of X with corresponding group T_F, the fixed point set of T_F is given by:*

$$M_X(\Lambda)^{T_F} = \bigcup_{G<X,\, T_G>T_F} \pi^{-1}(G)$$

where G is a face of X.

PROOF. First we will show that $\pi^{-1}(G) \subset M_X(\Lambda)^{T_F}$, for each face G for which $T_G > T_F$. Let $[t, x] \in \pi^{-1}(G)$. Then $x \in G$, which implies that $T_{F(x)} > T_G > T_F$. For $t' \in T_F$, $t'[t, x] = [t't, x]$. But $t't \cdot t^{-1} = t' \in T_F < T_{F(x)}$, which implies $[t't, x] = [t, x]$. Thus $[t, x] \in M_X(\Lambda)^{T_F}$.

For the inverse inclusion, let $[t, x]$ be fixed by T_F. Then, for $t' \in T_F$,

$$t'[t, x] = [t't, x] = [t, x] \Rightarrow t't \cdot t^{-1} = t' \in T_{F(x)} \Rightarrow T_F < T_{F(x)}.$$

So $[x, t] \in \pi^{-1}(T_{F(x)})$ with $T_F < T_{F(x)}$, completing the proof.

COROLLARY 3.2. *Let X be a nice manifold with corners and C a connected component $M_X(\Lambda)^{T_F}$. Then there is an element in C that is fixed by T.*

PROOF. Since $\pi^{-1}(F)$ is connected for each face F, Lemma 3.1 implies that C contains a space of the form $\pi^{-1}(F)$ with $F < X$. Since F contains 0-faces, C contains global fixed points.

The following results compare a locally standard torus manifold with its canonical model [24, Section 5]. In [24], Lemma 5.2 and Theorem 5.5, it is shown that the two manifolds M^{2n} and $M_X(\Lambda)$ are T-homeomorphic, with a homeomorphism covering the identity on X, if and only if a class $e(M^{2n}, X) \in \check{\mathrm{H}}^1(X, \mathscr{S}_{(X,\Lambda)})$, called the *Euler class*, vanishes. Here the cohomology theory is Čech cohomology with coefficients the sheaf of local sections of the quotient map $q\colon M_X(\Lambda) \to X$.

LEMMA 3.3 ([24]). *Let M^{2n} be a locally standard torus manifold over the nice manifold with corners X and $M_X(\Lambda)$ the canonical model associated to the action. Then the following are equivalent:*

(1) *There is a T-homeomorphism $h: M^{2n} \to M_X(\Lambda)$ covering the identity on X.*

(2) *The orbit map $\pi: M^{2n} \to X$ admits a section.*

(3) *The Euler class $e(X, M^{2n}) \in \check{\mathrm{H}}^1(X, \mathcal{S}_{(X,\Lambda)})$ vanishes.*

If any of the conditions above hold we say that the pair (M^{2n}, X) splits.

REMARK 3.4.

(1) Quasitoric manifolds split [4].

(2) If M^{2n} is a smooth locally standard torus manifold and $H^2(X, \mathbb{Z}) = 0$, then (M^{2n}, X) splits [10].

(3) The above two results are based on an explicit construction of a section of the quotient map. The construction is based on a procedure called "blowing-up the singular strata" that was developed in Lemma 1.4 in [4]. That procedure uses the smoothness assumption, among other things, and it can not be applied directly to the topological case.

(4) In [24], the above result was stated for the more general class of manifolds that admit a local torus action.

We would like to thank the referee for pointing out to us the following.

COROLLARY 3.5. *Let M^{2n} be a locally standard manifold over X. If $H^2(X; \mathbb{Z}^n) = 0$, then the pair (M^{2n}, X) splits. In particular, if X is contractible then the pair (M^{2n}, X) splits.*

PROOF. The definition of the Euler class $e(X, M^{2n})$ [24, Section 5] shows that there is an one-to-one correspondence between elements of $\check{\mathrm{H}}^1(X, \mathcal{S}_{(X,\Lambda)})$ and principal T^n-bundles over X. But such bundles are classified by elements of $H^2(X; \mathbb{Z}^n)$. Then the assumption implies that $e(X, M^{2n}) = 0$ and Lemma 3.3 implies that the pair (M^{2n}, X) splits.

Now we investigate the natural properties of the above construction.

DEFINITION 3.6. Let $\phi: Y \to X$ be a map between n-manifolds with corners.

(1) ϕ is called *skeletal* if it preserves skeleta, i.e., $\phi(Y^{(k)}) \subset X^{(k)}$.

(2) ϕ is called *face preserving* if, for each face F of Y, $\phi(F)$ is a face of X.

REMARK 3.7.

(1) Similarly, a homotopy $\phi_t: Y \to X$, is called skeletal (face preserving) if the map at each level is skeletal (face preserving).

(2) Notice that face-preserving maps or homotopies are skeletal.

(3) Let $\sigma: T \to T$ be a continuous automorphism. Let $f: N^{2n} \to M^{2n}$ be a σ-homotopy equivalence with N^{2n} (respectively M^{2n}) a locally standard torus manifold over Y (respectively X). Then f induces a skeletal homotopy equivalence $\phi: Y \to X$.

PROPOSITION 3.8. *Let* (X, Λ) *and* (Y, Λ') *be two characteristic pairs and* $\phi: Y \to X$ *be a skeletal homotopy equivalence with* X *a homotopy polytope. Then* ϕ *is a face-preserving homotopy equivalence.*

PROOF. We will show that ϕ is face preserving. So each level of the skeletal homotopy will be face preserving, completing the proof.

We use induction on the dimension of the faces. The inductive statement is the following:

ϕ induces a bijection between the sets of k-faces of Y and X.

Since ϕ induces a homotopy equivalence when restricted to the 0-skeleta, the statement is obviously true for the 0-faces, which are points. We assume that the statement is true for ϕ restricted to $Y^{(k-1)}$. That means that ϕ induces a bijection between the ℓ-faces of Y and X, for every $\ell = 0, \ldots, k-1$. Let G' be a k-face of Y. Set $p = n - k$. Then $\operatorname{rank}(T_{G'}) = p$. That is because G' is a component of the intersection of p facets. Also $\phi(G') \subset X^{(k)}$, since ϕ is skeletal.

If $\phi(G') \subset X^{(k-1)}$, then, by continuity, at least two $(k-1)$-subfaces of G' will map to the same $(k-1)$-face of X, contradicting the induction hypothesis. Thus $\phi(G') \not\subset X^{(k-1)}$.

Next we assume that $\phi(G')$ intersects at least two k-faces of X. Since ϕ is continuous, there are two k-faces, G_1 and G_2 which intersect in a $(k-1)$-face G so that $\phi(G')$ intersects the relative interiors of all three faces G_1, G_2 and G. By rearranging the order of facets if necessary, we have that

$$G_1 = F_1 \cap \cdots \cap F_p, \quad G_2 = F_2 \cap \cdots \cap F_{p+1}, \quad G = F_1 \cap \cdots \cap F_{p+1}.$$

Then $T_{G_1} = T_{F_1} \times \cdots \times T_{F_p}$, $T_{G_2} = T_{F_2} \times \cdots \times T_{F_{p+1}}$ and $T_G = T_{F_1} \times \cdots \times T_{F_{p+1}}$. If $T_{G_1} = T_{G_2}$ then $T_G = T_{G_1}$, a contradiction to Property (*) because the map $T_{F_1} \times \cdots \times T_{F_{p+1}} \to T$ is not an injection. Thus $T_{G_1} \neq T_{G_2}$. Then $T_{G'} \leq T_{G_1} \cap T_{G_2}$ and $p = \operatorname{rank}(T_{G'}) \leq \operatorname{rank}(T_{G_1 \cap G_2}) < p$. That gives a contradiction. Thus the image of G' is a face of X. That completes the proof.

COROLLARY 3.9. *The assumptions of Proposition 3.8 imply that* Y *is a homotopy polytope.*

PROOF. From Proposition 3.8, ϕ is a face preserving homotopy equivalence. That implies that Y is a nice manifolds with corners. Since every face of X

is the intersection of a set of facets, the same will be true for Y. Thus every preface of Y is a face. Since every face of X is contractible, the same will be true for the faces of Y. Thus Y is a homotopy polytope.

PROPOSITION 3.10. *Let (X, Λ) and (Y, Λ') be two characteristic pairs and $\sigma: T \to T$ be a continuous automorphism. Let also $\phi: Y \to X$ be a face-preserving map that satisfies $\sigma(T_{Y_i}) < T_{\phi(Y_i)}$ for each facet Y_i of Y. Then ϕ induces a σ-equivariant map $\phi_*: M_Y(\Lambda') \to M_X(\Lambda)$.*

PROOF. Define that map ϕ_* by equivariantly extending the map ϕ:

$$\phi_*: M_Y(\Lambda') \to M_X(\Lambda), \quad \phi_*([t, y]) = [\sigma(t), \phi(y)].$$

We need to show that the map is well-defined. Let $[t, y] = [t', y]$ in $M_Y(\Lambda')$. Then $t^{-1}t' \in T_{F'(y)}$. Also, $F'(y)$ is a component of the intersection of facets $Y_{i_1} \cap \cdots \cap Y_{i_m}$. So, $\phi(y)$ belongs to a component of the intersection of facets $\phi(Y_{i_1}) \cap \cdots \cap \phi(Y_{i_m}) = X'$. Since the map ϕ is face-preserving, there are facets X_i, $i = 1, \cdots, s$, of X such that $F(\phi(y))$ is a component of the intersection of faces $X' \cap X_1 \cap \cdots \cap X_s$. Thus $T_{X'} < T_{F(\phi(y))}$. Since $\sigma(T_{Y_i}) < T_{\phi(Y_i)}$, we have that $T_{F(\phi(y))} > T_{X'} > \sigma(T_{F'(y)})$. Therefore $\sigma(t^{-1}t') \in T_{F(\phi(y))}$ and $[\sigma(t), \phi(y)] = [\sigma(t'), \phi(y)]$ in $M_X(\Lambda)$. From the construction, the map is obviously σ-equivariant.

REMARK 3.11. In fact, the assumptions of Proposition 3.10 imply that $\sigma(T_{Y_i}) = T_{\phi(Y_i)}$ for each facet Y_i of Y. Notice that both $\sigma(T_{Y_i})$ and $T_{\phi(Y_i)}$ are subgroups of T isomorphic to S^1 and $\sigma(T_{Y_i}) < T_{\phi(Y_i)}$. Therefore they must be equal.

COROLLARY 3.12. *Let $\phi_s: Y \to X$ be face-preserving homotopies so that $\sigma(T_{Y_i}) < T_{\phi_s(Y_i)}$, for each $s \in [0, 1]$ and for each facet Y_i of Y. Then $\phi_{0,*} \simeq_\sigma \phi_{1,*}$.*

We now investigate the reverse construction.

PROPOSITION 3.13. *Let (X, Λ) and (Y, Λ') be two characteristic pairs such that X and Y are homotopy polytopes. Let $f: M_Y(\Lambda') \to M_X(\Lambda)$ be a σ-equivariant homotopy equivalence for some continuous automorphism $\sigma: T \to T$. Then*

(1) *the map $\phi: Y \to X$ induced on the quotients is a face-preserving homotopy equivalence,*
(2) *$\sigma(T_{F'}) = T_{\phi(F')}$ for each facet F' of Y,*
(3) *there is a σ-equivariant homotopy such that $f \simeq_\sigma \phi_*$.*

PROOF. The equivariance implies that ϕ is skeletal. Then Proposition 3.8 shows that the map ϕ is face preserving homotopy equivalence.

For (2), let Y_i be a facet of Y and T_{Y_i} its isotropy group. Then, equivariance again, implies that the isotropy group of $\phi(Y_i)$ contains $\sigma(T_{X_i})$. As in Remark 3.11, we see that $\sigma(T_{Y_i}) = T_{\phi(Y_i)}$. Let F' be a face of Y. Then $F' = Y_1 \cap \cdots \cap Y_m$ as an intersection of facets. Since ϕ induces a bijection on faces, $\phi(F') = \phi(Y_1) \cap \cdots \cap \phi(Y_m)$, as an intersection of facets. Then

$$\begin{aligned}\sigma(T_{F'}) &= \sigma(T_{Y_1} \times \cdots \times T_{Y_m}) = \sigma(T_{Y_1}) \times \cdots \times \sigma(T_{Y_m})\\ &= T_{\phi(Y_1)} \times \cdots \times T_{\phi(Y_m)} = T_{\phi(F')}.\end{aligned}$$

For (3), notice that the map f induces a map

$$f_Y\colon Y \to M_X(\Lambda), \quad \text{with } f_Y(y) = f([1, y]) = [t_y, \phi(y)],$$

for some $t_y \in T$.

For each face F' of Y, we write $F = \phi(F')$ and thus $T_F = \sigma(T_{F'})$. The restriction of f is a homotopy equivalence on fix point sets:

$$f| = f^{T_{F'}}\colon M_Y(\Lambda')^{T_{F'}} \to M_X(\Lambda)^{\sigma(T_{F'})} = M_X(\Lambda)^{T_F}$$

Write

$$f_{F'}\colon F' \xrightarrow{\iota_{F'}} M_Y(\Lambda')^{T'_F} \xrightarrow{f^{T_{F'}}} M_X(\Lambda)^{T_F}$$

where $\iota_{F'}(y) = [1, y]$. Explicitly, for $y \in F'$,

$$f_{F'}(y) = f^{T_{F'}} \circ \iota_{F'}(y) = f^{T_{F'}}([1, y]) = [t_y, \phi(y)].$$

Thus $f_{F'} = f_Y|F'$. Since $F = \phi(F')$ and $y \in F'$, we have that $\phi(y) \in F$. Also, define

$$\phi_{F'}\colon F' \xrightarrow{\iota_{F'}} M_Y(\Lambda')^{T'_F} \xrightarrow{\phi_*^{T_{F'}}} M_X(\Lambda)^{T_F}, \quad \phi_{F'}(y) = [1, \phi(y)]$$

Let $y_0 \in F'$ be a base point. Since F' is contractible, we choose a contracting homotopy c'_s starting from the identity on F' and ending to the constant map at y_0. Similarly, choose a contracting homotopy c_s from the identity on F to the constant map to $\phi(y_0)$. Set $W_F = T/T_F$. Since W_F is connected, we choose a path β in W_F with $\beta(0) = t_{y_0}T_F$ and $\beta(1) = T_F$. Define a homotopy $\chi_{F'}\colon F' \times I \to M_X(\Lambda)^{T_F}$, as follows

$$\chi_{F'}(y, s) = \begin{cases} f([1, c'_{2s}(y)]), & 0 \le s \le \frac{1}{2},\\ [\bar{\beta}(2s-1), c_{2-2s}(\phi(y))], & \frac{1}{2} \le s \le 1,\end{cases}$$

with $\bar{\beta}(2s-1)$ a coset representative of $\beta(2s-1)$.

Notice that

(1) $\chi_{F'}$ is well defined:
  (a) Let $\bar{\beta}_i(2s-1) \in T$, $i = 1, 2$, be two elements representing the same coset $T_F\beta(2s-1)$. Then, there is $t \in T_F$ such that $\bar{\beta}_1(2s-1) = t\bar{\beta}_2(2s-1)$. But

$$\begin{aligned} c_{2-2s}(\phi(y)) \in F &\Rightarrow F(c_{2-2s}(\phi(y))) \leq F \\ &\Rightarrow T_{F(c_{2-2s}(\phi(y)))} \geq T_F \\ &\Rightarrow t \in T_{F(c_{2-2s}(\phi(y)))}. \end{aligned}$$

  Therefore $\bar{\beta}_1(2s-1)(\bar{\beta}_2(2s-1))^{-1} = t \in T_{F(c_{2-2s}(\phi(y)))}$ and so

$$[\bar{\beta}_1(2s-1), c_{2-2s}(\phi(y))] = [\bar{\beta}_2(2s-1), c_{2-2s}(\phi(y))]$$

  from the definition. Hence the homotopy does not depend on the choice of the representative of $\beta(2s-1)$ in W_F.
  (b) For $s = 1/2$, the two branches of the function read:
    (i) $f([1, c_1'(y)]) = f([1, y_0]) = [t_{y_0}, \phi(y_0)]$,
    (ii) $[\bar{\beta}(0), c_1(\phi(y))] = [t_{y_0}, \phi(y_0)]$.
(2) $\chi_{F'}(y, 0) = f([1, y]) = [t_y, \phi(y)] = f_{F'}(y)$.
(3) $\chi_{F'}(y, 1) = [\bar{\beta}(1), \phi(y)] = [1, \phi(y)] = \phi_{F'}(y)$

For each face F', we will construct a homotopy $h_{F'}\colon F' \times [0, 1] \to M_X(\Lambda)^{T_F}$ such that:

(1) $h_{F'}$ is a homotopy from $f_{F'}$ to $\phi_{F'}$,
(2) for G' a subface of codimension 1 of F' (denoted $G' < F'$), the restriction of $h_{F'}$ to G' has the form:

$$h_{F'}|G'(y, s) = \begin{cases} h_{G'}(y, 2s), & 0 \leq s \leq \frac{1}{2}, \\ [1, \phi(y)], & \frac{1}{2} \leq s \leq 1. \end{cases}$$

We use the notation $h_{F'}|G' = h_G' * \phi_*$ for the concatenation of homotopies as above.

The construction is done inductively. For a 0-cell v', $\mathrm{Im}(f_{v'}) = \{[1, \phi(v')]\}$. So the homotopy on the 0-skeleton is the stationary homotopy. Let F' be an 1-cell. Then $\chi_{F'}$ induces a homotopy from $f_{F'}$ to $\phi_{F'}$. The homotopy extension property implies that there is a homotopy $h_{F'}$ between $f_{F'}$ and $\phi_{F'}$, rel$(\partial F')$.

Let F' be a k-face, $k > 1$. We assume that the homotopy has been defined on each subface of F'. The second property of the homotopies $h_{G'}$, for $G' < F'$, allows the assembly of the homotopies $h_{G'}$ in order to construct a homotopy

$h_{\partial F'}$ on $\partial F'$. The homotopy $h_{\partial F'}$ has the property that, for each $(k-1)$-dimensional subface $G' < F'$, $h_{\partial F'}|G' = h_{G'}$. Notice that, for each $G' < F'$, we have $G = \phi(G') < F$, $T_G > T_F$ and $M_X(\Lambda)^{T_G} < M_X(\Lambda)^{T_F}$. Thus, for each $(k-1)$-dimensional subface G', there is a homotopy

$$G' \times I \xrightarrow{h_{G'}} M_X(\Lambda)^{T_G} \xrightarrow{j_G} M_X(\Lambda)^{T_F},$$

where j_G is the inclusion map. That means that the homotopy $h_{\partial F'}$ induces a homotopy (also denoted $h_{\partial F'}$) $h_{\partial F'}\colon \partial F' \times I \to M_X(\Lambda)^{T_F}$. Using the homotopy extension property, we extend $h_{\partial F'}$ to a homotopy $g_{F'}\colon F' \times [0,1] \to M_X(\Lambda)^{T_F}$ such that

(1) $g_{F'}(y,0) = f_{F'}(y)$,
(2) for each $G' < F'$, a face of F' of codimension 1, $g_{F'}|G' = h_{G'}$.

Set $g_{F',1} = g_{F'}(-,1)$. Since $g_{F'}$ is the homotopy between $f_{F'}$ and $g_{F',1}$ and $\chi_{F'}$ is the homotopy between $f_{F'}$ and $\phi_{F'}$, there is a homotopy such that $g_{F',1} \simeq \phi_{F'}$. Also, for $y \in \partial F'$, y belongs to a $(k-1)$-dimensional subface of F' and

$$g_{F',1}(y) = g_{F'}(y,1) = h_{\partial F'}(y,1) = [1,\phi(y)] = \phi_{F'}(y).$$

Thus, there is a homotopy $\psi_{F'}$ such that $g_{F',1} \simeq \phi_{F'}$, rel$\partial F'$. Define the homotopy $h_{F'} = g_{F'} * \psi_{F'}$, the concatenation of the two homotopies. Then

(1) $h_{F'}(y,0) = g_{F'}(y,0) = f_{F'}(y)$,
(2) $h_{F'}(y,1) = \psi_{F'}(y,1) = \phi_{F'}(y)$,
(3) if $y \in \partial F'$, then
   (a) For $0 \le s \le 1/2$, $h_{F'}(y,s) = g_{F'}(y,2s) = h_{\partial F'}(y,2s)$,
   (b) For $1/2 \le s \le 1$, $h_{F'}(y,s) = \psi_{F'}(y,2s) = \phi_{F'}(y) = [1,\phi(y)]$.

Thus, $h_{F'}$ satisfies all the conditions required. Working inductively we get a homotopy $h\colon Y \times I \to M_X(\Lambda)$ such that

(1) $h(y,0) = f_Y(y) = f([1,y]) = [t_y,\phi(y)]$,
(2) $h(y,1) = \phi_*([1,y]) = [1,\phi(y)]$,
(3) for each face F' of Y, $\mathrm{Im}(h|F') \subset M_X(\Lambda)^{T_F}$.

Define $H\colon M_Y(\Lambda') \times I \to M_X(\Lambda)$, $H([t,y],s) = \sigma(t)h(y,s)$. Then H is the required homotopy between f and ϕ_*.

4. Rigidity

As before, we set $T = T^n$. Let M^{2n} be a locally standard closed torus manifold with $X = M^{2n}/T$ the corresponding nice n-manifold with corners. In this section, we assume that X is a homotopy polytope so that all the faces of X (and X itself) are contractible manifolds. Corollary 3.5 implies that the pair (M^{2n}, X) splits and thus there is a T-homeomorphism $M^{2n} \cong_T M_X(\Lambda)$ covering the identity on X. Here Λ is the characteristic map induced by the T-action on M^{2n}. Notice that the T action on $M_X(\Lambda)$ is effective and its isotropy groups are subtori of T. Thus the same is true for M^{2n}.

DEFINITION 4.1. A *locally linear T-action* on a manifold N^{2n} is a T-action so that every point y of N^{2n}, with isotropy group T_y, admits a T_y-slice, that is an orthogonal linear representation of T_y [1, p. 171].

REMARK 4.2. Locally standard torus manifolds are locally linear T-manifolds.

Let N^{2n} be a locally linear closed T-manifold. Let $f\colon N^{2n} \to M^{2n}$ be a T-equivariant homotopy equivalence with T-homotopy inverse g.

LEMMA 4.3. *The action of T on N^{2n} is effective.*

PROOF. We assume that is not the case. So there is some $t \in T$ that fixes N^{2n} pointwise. Let $G = \langle t \rangle$. Then $N^G = N^{2n} \simeq M^G$ since f is an equivariant homotopy equivalence. But M^G is a closed proper submanifold of M^{2n}, because the action on M^{2n} is effective. Thus $\dim(N^G) = \dim(M^G) < \dim(M^{2n}) = \dim(N^{2n})$, a contradiction.

LEMMA 4.4. *The non-trivial isotropy subgroups of N^{2n} are subtori of T.*

PROOF. Let y be in N^{2n} with isotropy group T_y that is not a subtorus of T. Since $N^{T_y} \simeq M^{T_y}$ and $N^{T_y} \neq \emptyset$, we have that $M^{T_y} \neq \emptyset$. Since the isotropy groups of M^{2n} are subtori, $M^{T'} = M^{T_y}$ for some subtorus T' that strictly contains T_y. But $y \in N^{T_y}$ and $y \notin N^{T'}$. Thus $N^{T_y} \supsetneq N^{T'} \simeq M^{T'}$. Since fixed point sets are closed submanifolds without boundary, we have that $\dim M^{T_y} = \dim N^{T_y} > \dim N^{T'} = \dim M^{T'}$. But this is a contradiction since $\dim M^{T_y} = \dim M^{T'}$.

COROLLARY 4.5. *The isotropy groups of N^{2n} and M^{2n} are the same.*

COROLLARY 4.6. *For each isotropy group T', each component of the fixed point set $N^{T'}$ contains a T-fixed point.*

PROOF. The subgroup T' is equal to T_F for some face $F < X$. Let C be a component of $N^{T'} = N^{T_F}$. The map f induces a homotopy equivalence $f^{T_F}\colon N^{T_F} \to M^{T_F}$. Thus, the restriction of f^{T_F} to C induces a homotopy equivalence between C and a component C' of M^{T_F}. Corollary 3.2 implies that C' has a T-fixed point. Therefore C also has a T-fixed point.

PROPOSITION 4.7. *The action of T on N^{2n} is locally standard. Furthermore, $N^{2n}/T = Y$ is a homotopy polytope.*

PROOF. Let $y \in N^{2n}$ with isotropy group T_y. Corollary 4.5 implies that $T_y = T_F$ for some face $F < X$. Let C be the component of N^{T_F} that contains y. Corollary 4.6 implies that there is a point $z \in C$ that is fixed by T. Since the action is locally linear, there is a linear slice around each point. Since the isotropy groups are subtori, the slices are well defined up to linear equivalence [19]. Notice that the tubes are smooth T-manifolds. Now we complete the proof as in [10, Theorem 4.1]. We start with a slice S at z. The slice S is a linear effective T-representation. If $y \in S$, then S is a locally standard neighborhood of y. In general, there is a path α that joins z to y. We use the same argument as in the end of the proof of Theorem 4.1 in [10], moving along the path. We start with a finite open cover of the image of α with tubes. Then, there is a tube τ that contains z. The tangent space $\mathscr{T}_z$ is an effective T-representation. Let y_0 be a point on the path that lies in the intersection of τ and another tube τ_0 at x_0. Then the tangent space at y_0 is a $\mathscr{T}_{y_0}$-representation, and this is the restriction of the T-representation $\mathscr{T}_z$ and the T_{x_0}-representation $\mathscr{T}_{x_0}$. Continuing this way we get a sequence of points y_i, $i = 0, 1, \ldots, k$, such that $y_k = y$ and the T_{y_i}-representation $\mathscr{T}_{y_i}$ is the restriction of the T-representation $\mathscr{T}_z$. That completes the proof of the first part.

The quotient Y is a nice n-manifold with corners and there is a skeletal homotopy equivalence $\phi\colon Y \to X$. Corollary 3.9 implies that Y is a homotopy polytope.

We denote by Λ' the characteristic function determined by the T-action on N^{2n}. Proposition 3.8 implies that the map f induces a face-preserving homotopy equivalence $\phi\colon Y \to X$. By Proposition 3.13, the map f is T-homotopic to ϕ_*.

We need a version of the Poincaré Conjecture. For an n-dimensional manifold with boundary $(M, \partial M)$ the relative structure set $\mathscr{S}(M, \partial M)$ is the set of equivalence classes of pairs (N, f) with N an n-dimensional manifold with boundary and $f\colon N \to M$ a homotopy equivalence such that $\partial f\colon \partial N \to \partial M$ is a homeomorphism.

For the following lemma, the structure set is defined as follows:

$$\mathscr{S}(M, \partial M) = \{f\colon (X^n, \partial X) \to (M, \partial M) \mid f \text{ a homotopy equivalence}, \\ f|_{\partial X} \text{ homeomorphism}\}/\sim$$

where the equivalence relation is given by homeomorphisms.

LEMMA 4.8. *Let $(M, \partial M)$ be a compact contractible n-manifold with boundary. Then the relative structure set $\mathcal{S}(M, \partial M) = *$, $n \neq 3$. If $n = 3$ and $M \subset S^3$, then $(M, \partial M) \cong (D^3, S^2)$.*

PROOF. For $n = 1, 2$ the result is obvious. For $n \geq 4$, there is the surgery exact sequence:

$$\cdots \to [(M \times I, \partial M \times I), (G/Top, *)] \to L_{n+1}(\mathbb{Z}) \\ \to \mathcal{S}(M, \partial M) \to [(M, \partial M), (G/Top, *)] \to L_n(\mathbb{Z})$$

(for $n \geq 5$ this is the classical surgery exact sequence [20], for $n = 4$ the result follows from the results of Freedmann [5], see also Kirby-Taylor [7, §7]. But $M/\partial M \cong S^n$ and $\mathcal{S}(S^n) = *$ for $n \geq 4$. We also have commutative diagrams

$$\begin{array}{ccc} [(M \times I, \partial M \times I), (G/Top, *)] & \longrightarrow & L_{n+1}(\mathbb{Z}) \\ \cong\downarrow & & \| \\ [(M/\partial M) \times I, G/Top] & \longrightarrow & L_{n+1}(\mathbb{Z}) \end{array}$$

and

$$\begin{array}{ccc} [(M, \partial M), (G/Top, *)] & \longrightarrow & L_n(\mathbb{Z}) \\ \cong\downarrow & & \| \\ [M/\partial M, G/Top] & \longrightarrow & L_n(\mathbb{Z}) \end{array}$$

So the vanishing of the structure set of the sphere implies that, in the exact sequence, the first map is onto and the last map is into. Thus $\mathcal{S}(M, \partial M) = *$.

For $n = 3$, the results of Perelman ([14], [16], [15]) imply that there are no fake disks and spheres in dimension 3. Thus $(M, \partial M) \cong (D^3, S^2)$ [17, Lemma 5.2] and [18, Proof of Theorem 3.10].

LEMMA 4.9. *Let X and Y be homotopy polyhedra. Then any face-preserving homotopy equivalence $\phi\colon Y \to X$ is face-preserving homotopic to a face-preserving homeomorphism.*

PROOF. We will use the method that was used in [12], [17], and [18] to show that ϕ is face-preserving homotopic to a face-preserving homeomorphism. We will construct a face-preserving homeomorphism by induction on faces. Notice that each closed face is homeomorphic to a contractible manifold with boundary.

We will use induction on the dimension of the faces. Since we have the same number of zero faces in the two homotopy polytopes, the restriction of ϕ to zero faces is a homeomorphism. Now, let F_1 be a face of Y and ∂F_1 its boundary. We assume that there is face-preserving homeomorphism $h_{\partial F_1}$ face-preserving homotopic to $\phi|_{\partial F_1}$. Using the homotopy extension property, there is a map $\phi' \colon F_1 \to F_2$ that is homotopic to $\phi|_{F_1}$ and it extends the map $h_{\partial F_1}$. Because all the maps and homotopies are face-preserving at the boundary, they are face-preserving in the closed face F_1. By Lemma 4.8, ϕ' is homotopic to a homeomorphism relative to the boundary. As before, all homotopies are face-preserving. Continuing this way, we get a face-preserving homeomorphism $h \colon Y \to X$ that is face-preserving homotopic to ϕ.

Lifting the maps and the homotopies to the canonical models, we have the following.

COROLLARY 4.10. *Let* (Y, Λ') *and* (X, Λ) *be characteristic pairs, with* X *and* Y *homotopy polyhedra. Let* $\phi \colon Y \to X$ *be a face-preserving homotopy equivalence. Then the induced map* $\phi_* \colon N_Y(\Lambda') \to M_X(\Lambda)$ *is* T*-homotopic to a* T*-homeomorphism.*

THEOREM 4.11 (Rigidity of Locally Standard Torus Manifolds). *Let* M^{2n} *be a closed locally standard torus manifold over an* n*-manifold with corners* X *and characteristic map* Λ. *We assume that* X *is a homotopy polytope and all the faces of* X (*and* X *itself*) *are contractible manifolds with corners. Let* N^{2n} *be a locally linear closed* T*-manifold and* $f \colon N^{2n} \to M^{2n}$ *a* T*-equivariant homotopy equivalence. Then* f *is* T*-homotopic to a* T*-homeomorphism.*

PROOF. From Proposition 4.7, the action of T on N^{2n} is locally standard. Then, Corollary 3.5 implies that the pairs (M^{2n}, X) and (N^{2n}, Y) split.

The map f induces a face-preserving map $\phi \colon Y \to X$. Let

$$\bar{f} \colon N_Y(\Lambda') \xrightarrow{\cong} N^{2n} \xrightarrow{f} M^{2n} \xrightarrow{\cong} M_X(\Lambda).$$

It is enough to show that $\bar{f}$ is T-homotopic to a T-homeomorphism. Notice that $\bar{f}$ also induces the map ϕ on the quotients. By Proposition 3.13, $\bar{f} \simeq_T \phi_*$, and, by Corollary 4.10, ϕ_* is T-homotopic to a T-homeomorphism h. Thus $\bar{f} \simeq_T \phi_* \simeq_T h$ and the last map is a T-homeomorphism.

REMARK 4.12. In [24], Theorem 6.2 provides a complete classification of locally standard torus manifolds. That classification applies to the above result. The difference is that the homeomorphism given in [24] it is not necessarily equivariantly homotopic to the original homotopy equivalence.

The following is an immediate consequence of Theorem 4.11.

COROLLARY 4.13. *Let* M^{2n} *be a quasitoric manifold. Let* N^{2n} *a locally linear* T*-manifold and* $f \colon N^{2n} \to M^{2n}$ *a* T*-homotopy equivalence. Then* f *is* T*-homotopic to a* T*-homeomorphism.*

Finally, a slightly more general result holds.

COROLLARY 4.14. *Let M^{2n} be a locally standard torus manifold over an n-manifold with corners X. We assume that M^{2n} satisfies the conditions of Theorem 4.11. Let $\sigma\colon T \to T$ be a continuous automorphism, N^{2n} be a locally linear T-manifold and $f\colon N^{2n} \to M^{2n}$ be a σ-equivariant homotopy equivalence. Then f is σ-homotopic to a σ-homeomorphism.*

An important class of quasitoric manifolds are the complex projective spaces. In most cases, products of these spaces do not have vanishing structure sets. But they are rigid as locally standard torus manifolds.

REFERENCES

1. Bredon, G. E., *Introduction to compact transformation groups*, Pure and Applied Mathematics, vol. 46, Academic Press, New York-London, 1972.
2. Bukhshtaber, V. M., and Panov, T. E., *Actions of tori, combinatorial topology and homological algebra*, Uspekhi Mat. Nauk 55 (2000), no. 5 (335), 3–106, preprint arXiv:math/0010073 [math.AT].
3. Davis, M. W., *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008.
4. Davis, M. W., and Januszkiewicz, T., *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62 (1991), no. 2, 417–451.
5. Freedman, M. H., *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982), no. 3, 357–453.
6. Fulton, W., *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993.
7. Kirby, R. C., and Taylor, L. R., *A survey of* 4*-manifolds through the eyes of surgery*, in "Surveys on surgery theory, Vol. 2", Ann. of Math. Stud., vol. 149, Princeton Univ. Press, Princeton, NJ, 2001, pp. 387–421.
8. Kuroki, S., *An Orlik-Raymond type classification of simply connected* 6*-dimensional torus manifolds with vanishing odd-degree cohomology*, Pacific J. Math. 280 (2016), no. 1, 89–114.
9. Masuda, M., *Equivariant cohomology distinguishes toric manifolds*, Adv. Math. 218 (2008), no. 6, 2005–2012.
10. Masuda, M., and Panov, T., *On the cohomology of torus manifolds*, Osaka J. Math. 43 (2006), no. 3, 711–746.
11. Masuda, M., and Suh, D. Y., *Classification problems of toric manifolds via topology*, in "Toric topology", Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 273–286.
12. Moussong, G., and Prassidis, S., *Equivariant rigidity theorems*, New York J. Math. 10 (2004), 151–167.
13. Oda, T., *Convex bodies and algebraic geometry: An introduction to the theory of toric varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 15, Springer-Verlag, Berlin, 1988.
14. Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, preprint arXiv:math/0211159 [math.DG], 2002.
15. Perelman, G., *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, preprint arXiv:math/0307245 [math.DG], 2003.

16. Perelman, G., *Ricci flow with surgery on three-manifolds*, preprint arXiv:math/0303109 [math.DG], 2003.
17. Prassidis, S., and Spieler, B., *Rigidity of Coxeter groups*, Trans. Amer. Math. Soc. 352 (2000), no. 6, 2619–2642.
18. Rosas, E., *Rigidity theorems for right angled reflection groups*, Trans. Amer. Math. Soc. 308 (1988), no. 2, 837–848.
19. Schultz, R., *On the topological classification of linear representations*, Topology 16 (1977), no. 3, 263–269.
20. Wall, C. T. C., *Surgery on compact manifolds*, London Mathematical Society Monographs, no. 1, Academic Press, London-New York, 1970.
21. Wiemeler, M., *Remarks on the classification of quasitoric manifolds up to equivariant homeomorphism*, Arch. Math. (Basel) 98 (2012), no. 1, 71–85.
22. Wiemeler, M., *Exotic torus manifolds and equivariant smooth structures on quasitoric manifolds*, Math. Z. 273 (2013), no. 3-4, 1063–1084.
23. Yoshida, T., *Locally standard torus fibrations*, in "Proceedings of 34th Symposium on Transformation Groups", Wing Co., Wakayama, 2007, pp. 107–118.
24. Yoshida, T., *Local torus actions modeled on the standard representation*, Adv. Math. 227 (2011), no. 5, 1914–1955.

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