PENCILS AND NETS ON CURVES ARISING FROM RANK 1 SHEAVES ON K3 SURFACES

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Abstract

Let S be a K3 surface, C a smooth curve on S with $\mathcal{O}_S(C)$ ample, and A a base-point free g_d^2 on C of small degree. We use Lazarsfeld-Mukai bundles to prove that A is cut out by the global sections of a rank 1 torsion-free sheaf $\mathcal{G}$ on S. Furthermore, we show that $c_1(\mathcal{G})$ with one exception is adapted to $\mathcal{O}_S(C)$ and satisfies $\mathrm{Cliff}(c_1(\mathcal{G})_{|C}) \leq \mathrm{Cliff}(A)$, thereby confirming a conjecture posed by Donagi and Morrison. We also show that the same methods can be used to give a simple proof of the conjecture in the g_d^1 case.

In the final section, we give an example of the mentioned exception where $h^0(C, c_1(\mathcal{G})_{|C})$ is dependent on the curve C in its linear system, thereby failing to be adapted to $\mathcal{O}_S(C)$.

1. Introduction

In the past 30 years, one central problem in the study of the existence of g_d^r's on smooth curves has been to find connections between sheaves on K3 surfaces S and linear systems on curves C lying on S. This started with Lazarsfeld [10] and Tyurin [13] independently introducing vector-bundles $\mathcal{E}_{C,A}$ on S, depending on a smooth curve C and base-point free complete linear system A on C, providing much information on the geometry of the curve C and existence of other linear systems on the curve.

These vector-bundle techniques have given grounds for many results. Among these, Green and Lazarsfeld [6] proved in 1987 that the Clifford index is constant for all smooth curves C in a linear system on a K3 surface, and that it is maximal (i.e., equal to $\lfloor (g-1)/2 \rfloor$) if the Picard group is generated by $\mathcal{O}_S(C)$. In 1995, Ciliberto and Pareschi [3] proved that the gonality is constant for all smooth curves in an ample linear system on a K3 surface. Knutsen [8] proved that this is also true for non-ample linear systems, except for one particular case. Furthermore, he proved that there exist only two examples of exceptional curves. Among other notable results, Aprodu and Farkas [1] proved in 2011 that the Green conjecture is satisfied for all smooth curves on K3 surfaces. They also found the exact dimension of the g_d^1's for the general curves in a linear system.

Lelli-Chiesa [12] proved a conjecture posed by Donagi and Morrison [4], in the case of K3 surfaces without (-2) curves, $d \leq g - 1$ and $\mathrm{Cliff}(A) = \mathrm{Cliff}(C)$. The conjecture is stated as follows:

CONJECTURE 1.1 (Donagi-Morrison [4]). *Suppose C is a smooth curve on a K3 surface S, and let A be a base-point free complete g_d^r on C such that $\rho(g, r, d) < 0$. Then there exists a line bundle D on S, adapted to $\mathcal{O}_S(C)$, such that $A \leq D_{|C}$ and $\mathrm{Cliff}(D_{|C}) \leq \mathrm{Cliff}(A)$.*

Here, the *Clifford index* of a line bundle A on a smooth curve C is defined as $\mathrm{Cliff}(A) := \deg(A) - 2(h^0(C, A) - 1)$. We also mention that $\mathrm{Cliff}(C) := \min\{\mathrm{Cliff}(A) \mid h^0(C, A), h^1(C, A) \geq 2\}$ (but where $\mathrm{Cliff}(C)$ is defined to be 0 for hyperelliptic curves of genus 2 or 3, and 1 for trigonal curves of genus 3). The value $\rho(g, r, d)$ is the *Brill-Noether number* and is defined as $\rho(g, r, d) := g - (r+1)(g - d + r)$. A line bundle D on S is said to be *adapted to the line bundle L* if

(i) $h^0(S, D) \geq 2$ and $h^0(S, L \otimes D^\vee) \geq 2$, and

(ii) $h^0(C, D_{|C})$ is independent of the curve $C \in |L|_s$,

where $|L|_s$ denotes smooth curves in $|L|$.

The conjecture was proved in [4] for the case of g_d^1's, and basically involved proving that c_1 of the cokernel of the maximal destabilising sequence of $\mathcal{E}_{C,A}$ satisfies the conditions of the line bundle D in the conjecture, with the exception of one special case.

Part of the conjecture was also proved by Lelli-Chiesa in [11] for the case of g_d^2's on curves on maximal gonality and Clifford dimension 1. There, the idea was to prove that the kernel of the maximal destabilising sequence of $\mathcal{E}_{C,A}$ can be assumed to be of rank 1, and that the determinant of the cokernel is the desired line bundle D of the conjecture.

In the proof of our result, we use similar ideas. The main result states that the divisors in base-point free complete g_d^2's for small d are equal to global sections of torsion-free sheaves of rank 1 on S restricted to C. The torsion-free sheaves arise naturally from a maximal destabilising sequence of $\mathcal{F}_{C,A} = \mathcal{E}_{C,A}^\vee$, and c_1 of these sheaves satisfy the conditions on D of the conjecture, similar to what is done in Donagi-Morrison's and Lelli-Chiesa's proofs.

Our main result is the following:

THEOREM 1.2. *Let S be any K3 surface, and let L be an ample line-bundle on S. If $C \in |L|$ is smooth and A is a base-point free complete g_d^2 on C satisfying $d \leq \frac{1}{6}L^2$, then the following is satisfied:*

(a) *there exist a linear system $|D|$ on S and a finite subscheme $\xi \subset S$ such that every divisor in $|A|$ is equal to an element in $|D \otimes \mathcal{I}_\xi|$ restricted to C, where $\mathcal{I}_\xi$ is the ideal sheaf of ξ.*

*Suppose furthermore that there do not exist an elliptic pencil E and a (-2)
curve Γ satisfying both conditions $A = E^{\otimes 2}_{|C}$ and $(L \otimes E^{\otimes(-2)}).\Gamma = -2$. Then:*

(b) *the line bundle D found in (a) is adapted to L; and*

(c) $\mathrm{Cliff}(D_{|C}) \leq \mathrm{Cliff}(A)$.

REMARK 1.3. In the case where there exist an elliptic pencil E and a (-2)
curve Γ satisfying $A = E^{\otimes 2}_{|C}$ and $(L \otimes E^{\otimes(-2)}).\Gamma = -2$, we can construct
examples where $h^0(C, E^{\otimes 2}_{|C})$ is dependent on the curve C in $|L|_s$. See Section 3.

The tools used in the proof of Theorem 1.2, can also be used to give a simple
proof of Donagi and Morrison's result for the g^1_d case [4, Theorem 5.1'].
Here, we avoid the special case that was considered in the original proof.
Furthermore, as in Theorem 1.2, we also here prove that all divisors in $|A|$
are equal to the restriction to C of the global sections of a rank-1 torsion-free
sheaf on S.

THEOREM 1.4. *Let $|L|$ be any base-point free linear system on a K3 surface
S. If $C \in |L|$ is smooth and A is a base-point free complete g^1_d on C satisfying
$\rho(g, 1, d) < 0$, then the following is satisfied:*

(a) *there exists a linear system $|D|$ on S and a finite subscheme $\xi \subset S$ such
that every divisor in $|A|$ is equal to an element in $|D \otimes \mathcal{I}_\xi|$ restricted to
C, where $\mathcal{I}_\xi$ is the ideal sheaf of ξ;*

(b) *the line bundle D found in (a) is adapted to L; and*

(c) $\mathrm{Cliff}(D_{|C}) \leq \mathrm{Cliff}(A)$.

We will be working in characteristic 0 throughout this paper.

2. Proof of theorem

The Lazarsfeld-Mukai vector bundles are defined as follows: given a smooth
curve C of genus g on S and a base-point free, complete g^r_d A on C, the
vector-bundle $\mathcal{F}_{C,A}$ on S is defined as the kernel of the evaluation morphism
$H^0(C, A) \otimes \mathcal{O}_S \to A \to 0$. The bundle has the following properties:

- $\mathrm{rk}(\mathcal{F}_{C,A}) = r + 1$,
- $\det(\mathcal{F}_{C,A}) = \mathcal{O}_S(-C)$,
- $c_2(\mathcal{F}_{C,A}) = d$,
- $h^0(S, \mathcal{F}_{C,A}) = h^1(S, \mathcal{F}_{C,A}) = 0$,
- $\chi(S, \mathcal{F}_{C,A}) = 2 - 2\rho(g, r, d)$, where $\rho(g, r, d) = g - (r+1)(g-d+r)$
  is the Brill-Noether number,
- the dual, $\mathcal{F}^\vee_{C,A}$, is globally generated away from a finite set.

Note that if $\rho(g, r, d) < 0$, then $2h^0(S, \mathscr{F}_{C,A} \otimes \mathscr{F}_{C,A}^\vee) \geq \chi(S, \mathscr{F}_{C,A}) \geq 4$, and so $\mathscr{F}_{C,A}$ is then non-simple, and hence non-stable.

We will for the remainder of this paper – except in the proof of Theorem 1.4 – assume that A is a base-point free, complete g_d^2 on a smooth curve $C \in |L|$ with L ample and $d \leq \frac{1}{6}L^2$. By [7, Theorem 3.4.1], $\mathscr{F}_{C,A}$ is then unstable, and there thus exists a maximal destabilising sequence

$$0 \longrightarrow M \longrightarrow \mathscr{F}_{C,A} \longrightarrow N \longrightarrow 0 \tag{1}$$

such that M is locally free, N is torsion-free and μ_L-semistable, and $c_1(M).L > -\operatorname{rk}(M)\frac{1}{3}L^2$.

In the statements that follow, we will also be needing the dualisation of this sequence, which is

$$0 \longrightarrow N^\vee \longrightarrow \mathscr{F}_{C,A}^\vee \longrightarrow \tilde{M} \longrightarrow 0, \tag{2}$$

where $\tilde{M}$ is torsion-free and satisfies $\tilde{M}^\vee = M$. Since $\mathscr{F}_{C,A}^\vee$ is globally generated away from a finite set, the same applies for $\tilde{M}$. This sequence is maximal destabilising for $\mathscr{F}_{C,A}^\vee$, and so $\tilde{M}$ must be μ_L-semistable.

The following lemma is needed for the proof of Proposition 2.2, which (among other things) states that we can assume the rank of M to be 2. This is the most important step for the proof of Theorem 1.2.

LEMMA 2.1. *Let A, C and L be as above, and consider the maximal destabilising sequence (1) of $\mathscr{F}_{C,A}$. If $\operatorname{rk}(M) = 1$, then $M.c_1(N) \geq 0$.*

PROOF. Suppose $M.c_1(N) = M^\vee.c_1(N^\vee) < 0$.
We dualise the sequence (1), yielding

$$0 \longrightarrow N^\vee \longrightarrow \mathscr{F}_{C,A}^\vee \longrightarrow M^\vee \otimes \mathscr{I}_\eta \longrightarrow 0,$$

where $\mathscr{I}_\eta$ is the ideal sheaf of a 0-dimensional subscheme η. Since $\mathscr{F}_{C,A}^\vee$ is globally generated away from a finite set, then so is $M^\vee$, and it follows that a sufficient condition for $M.c_1(N) = M^\vee.c_1(N^\vee)$ to be ≥ 0 is that $h^0(S, c_1(N^\vee)) \geq 1$.

Now, since $M.L > -\frac{1}{3}L^2$ and $M \otimes c_1(N) \cong L^\vee$, then $c_1(N).L < -\frac{2}{3}L^2 < 0$, and so it suffices to show that $c_1(N)^2 \geq 0$, since it then follows that either $c_1(N)$ or $c_1(N^\vee)$ must be effective, and we see that it must be $c_1(N^\vee)$.

Now, since we have found that $c_1(N).L < -\frac{2}{3}L^2$, then $c_1(N^\vee).L > \frac{2}{3}L^2$. This gives us the inequality $c_1(N^\vee)^2 + c_1(N^\vee).M^\vee = c_1(N^\vee).L > \frac{2}{3}L^2$. And since we are assuming that $c_1(N^\vee).M^\vee < 0$, it follows that $c_1(N^\vee)^2 > 0$.

The result follows.

PROPOSITION 2.2. *Let A, C and L be as above, and let (1) be a maximal destabilising sequence of $\mathscr{F}_{C,A}$. Then* $\mathrm{rk}(M) = 2$; $c_1(M)^2 \geq 0$; $c_2(\tilde{M}) \geq 0$, *where $\tilde{M}$ is as in (2); and* $c_2(M) \geq 0$.

PROOF. Suppose $\mathrm{rk}(M) = 1$. Then N is semistable of rank 2, and so by [7, Theorem 3.4.1], $c_2(N) \geq \frac{1}{4}c_1(N)^2$. Furthermore, $M.L > -\frac{1}{3}L^2$, and so $c_1(N).L < -\frac{2}{3}L^2$. From (1), we hence get $c_2(\mathscr{F}_{C,A}) \geq M.c_1(N) + \frac{1}{4}c_1(N)^2 = \frac{1}{4}c_1(N)(c_1(N) + 4M) = \frac{1}{4}c_1(N)(-L + 3M) > \frac{2}{12}L^2 + \frac{3}{4}c_1(N).M$, and by Lemma 2.1, this is $\geq \frac{1}{6}L^2$. Since $c_2(\mathscr{F}_{C,A}) = \deg(A)$, which was assumed to be $\leq \frac{1}{6}L^2$, this gives the desired contradiction.

To prove the first two inequalities of the statement, we consider (2) and note that since $\tilde{M}$ is globally generated away from a finite set, then the same must apply for $c_1(\tilde{M}) = c_1(M^\vee)$, and so $c_1(M)^2 = c_1(M^\vee)^2 \geq 0$. By [7, Theorem 3.4.1], we must have $c_2(\tilde{M}) \geq 0$ as a consequence.

We now prove the last statement. Since $M^\vee$ is globally generated away from a finite set, we can inject an effective line-bundle D_1 into $M^\vee$, assume that the injection is saturated, and get

$$0 \longrightarrow D_1 \longrightarrow M^\vee \longrightarrow D_2 \otimes \mathscr{I}_\eta \longrightarrow 0,$$

where η is a possibly empty zero-dimensional subscheme and D_2 a line-bundle. Since $M^\vee$ is globally generated away from a finite set, then D_2 is also globally generated (actually everywhere since it is base-component free). But then, $D_1.D_2 \geq 0$, and dualising, we see that $c_2(M) \geq 0$.

In the proof of part (c) of Theorem 1.2, we will need the following result.

PROPOSITION 2.3. *Suppose D_1 and D_2 are two divisors on a K3 surface, and suppose $D_2^2 > 0$. Then $D_1^2 D_2^2 \leq (D_1.D_2)^2$, with equality if and only if $(D_1.D_2)D_1 \sim D_1^2 D_2$.*

PROOF. This follows from the Hodge Index Theorem (see e.g. [2, Corollary 2.16]) and [5, Chapter 1, Exercise 10], and using that numeric and linear equivalence are the same for divisors on a K3 surface.

We now give the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. We begin by proving part (a) of the theorem.

Let A, C and L be as in the theorem, and let $\mathscr{F}_{C,A}$ be the associated Lazarsfeld-Mukai bundle. Since $\deg(A) \leq \frac{1}{6}L^2$, it follows from [7, Theorem 3.4.1] that $\mathscr{F}_{C,A}$ is unstable, and so we obtain a maximal destabilising sequence (1).

The injection $M \hookrightarrow \mathscr{F}_{C,A}$ can be composed with $\mathscr{F}_{C,A} \hookrightarrow \mathcal{O}_S^{\oplus 3}$, yielding the following diagram, where $\mathscr{G}$ is the cokernel:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathcal{O}_S^{\oplus 3} & \xrightarrow{\tilde{\mathrm{ev}}} & \mathscr{G}^\phi & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}_{C,A} & \longrightarrow & \mathcal{O}_S^{\oplus 3} & \xrightarrow{\mathrm{ev}} & A & \longrightarrow & 0
\end{array}
\tag{3}
$$

By the snake lemma, $\ker(\phi) \cong N$, and since any torsion element of $\mathscr{G}$ must map to 0 in A and N is torsion-free, it follows that $\mathscr{G}$ is torsion-free. Since $\mathrm{rk}(M) = 2$, by Proposition 2.2, we have $\mathrm{rk}(\mathscr{G}) = 1$, and so it follows that $\mathscr{G} = D \otimes \mathscr{I}_\xi$, where $D = c_1(M^\vee)$ and $\mathscr{I}_\xi$ is the ideal sheaf of a possibly empty finite subscheme ξ. Furthermore, ϕ is injective on global sections, since $h^0(S, N) = 0$ (proof: If $h^0(S, N) > 0$, then – since $\mathrm{rk}(N) = 1$ – we can write $N \cong N' \otimes \mathscr{I}_\zeta$ where $\mathscr{I}_\zeta$ is the ideal sheaf of a possibly empty finite subscheme ζ, and where N' is an effective line bundle. But then $c_1(N) = N'.L \geq 0$, contradicting (1) being a maximal destabilising sequence).

It is clear that $h^0(S, M) = 0$ since $h^0(S, \mathscr{F}_{C,A}) = 0$, and so $h^0(S, \mathscr{G}) \geq 3 = h^0(C, A)$. Since ϕ is injective on global sections, this means that $h^0(S, \mathscr{G}) = 3$, and so each global section of A comes from a unique global section of $\mathscr{G}$. The map ϕ is an element of $\mathrm{Hom}(D \otimes \mathscr{I}_\xi, A) = \mathrm{Hom}(\mathscr{I}_\xi, A \otimes D^\vee)$, implying that either $A \cong D_{|C} \otimes \mathscr{I}_{\xi'}$, with $\xi' = \xi \cap C$, or $h^0(C, A) > h^0(C, D \otimes \mathscr{I}_{\xi'})$. However, we have $h^0(C, A) = h^0(S, D \otimes \mathscr{I}_\xi)$, and having $h^0(S, D \otimes \mathscr{I}_\xi) > h^0(C, D \otimes \mathscr{I}_{\xi'})$ would imply that $h^0(S, D \otimes \mathcal{O}_S(-C)) > 0$, which contradicts that $c_1(M).L > -\frac{2}{3}L^2$.

We conclude that $A \cong D_{|C} \otimes \mathscr{I}_{\xi'}$, and that every divisor in $|A|$ comes from restricting divisors in $|D|$ vanishing on ξ.

Proof of part (b): We recall that D is by definition adapted to L if $h^0(S, D) \geq 2$, $h^0(S, L \otimes D^\vee) \geq 2$, and $h^0(C, D_{|C})$ is independent of the curve C in $|L|_s$.

We already know that $h^0(S, D \otimes \mathscr{I}_\xi) = 3$, so the first condition is clear. To show that $h^0(S, L \otimes D^\vee) \geq 2$, note that $D \otimes L^\vee \cong c_1(N)$, which cannot be effective since $c_1(N).L < -\frac{1}{3}L^2$. It thus suffices, by Riemann-Roch, to show that $c_1(N^\vee)^2 \geq 0$. We have $c_1(N^\vee)^2 = (L - c_1(M^\vee)).c_1(N^\vee) = L.c_1(N^\vee) - c_1(M^\vee).c_1(N^\vee) > \frac{1}{3}L^2 - c_1(N^\vee).c_1(M^\vee)$. Sequence (2) gives us $\frac{1}{6}L^2 \geq c_2(\mathscr{F}_{C,A}^\vee) = c_1(M^\vee).c_1(N^\vee) + c_2(\tilde{M}) \geq c_1(M^\vee).c_1(N^\vee)$ (using Proposition 2.2 on $c_2(\tilde{M})$, and that $c_2(N^\vee) = 0$ since it is locally free of rank 1), and so we can conclude that $c_1(N^\vee)^2$ is nonnegative (actually, it is $\geq \frac{1}{6}L^2$).

We now show that $h^0(C, D_{|C})$ is independent of the curve C in $|L|_s$. By taking cohomology of the sequence $0 \to D \otimes L^\vee \to D \to D_{|C} \to 0$, we see that it suffices to show that $h^1(S, D) = 0$. We saw in Proposition 2.2 that $c_1(M)^2$, which is the same as D^2, is nonnegative, and so $h^1(S, D) > 0$ if and

only if there exists a -2-curve Γ such that $\Gamma.D < 0$ or $D = \mathcal{O}_S(nE)$ for some positive integer n and where E is an elliptic curve (see [9, Theorem]). Since $\tilde{M}$ is globally generated away from a finite set, then so is D (it is actually base-point free, since it is on a K3 surface and has no base-components), and so no -2-curve can intersect D negatively. In order to prove that $h^1(S, D) = 0$, it therefore suffices to prove that $D^2 > 0$.

To prove that $D^2 > 0$, the top row of (3) shows that $0 = c_1(M).c_1(\mathcal{G}) + c_2(M) + c_2(\mathcal{G}) = -c_1(M)^2 + c_2(M) + c_2(\mathcal{G})$. Since $c_2(M) \geq 0$ by Proposition 2.2, then $c_1(M)^2 \geq c_2(\mathcal{G})$. We obviously have $c_2(\mathcal{G}) \geq 0$, and we have $c_1(M)^2 = 0$ only if $c_2(\mathcal{G}) = 0$, and hence only if $\mathcal{G} = D$, with $h^0(S, D) = 3$. However, in this case, $A = D_{|C}$, and we clearly see that $D = E^{\otimes 2}$ where E is an elliptic pencil.

We prove that $h^0(C', E^{\otimes 2}_{|C'}) = 3$ for all $C' \in |L|_s$ under the conditions of the theorem, given that there exists a curve $C \in |L|_s$ where $h^0(C, E^{\otimes 2}_{|C}) = 3$. Consider the exact sequence

$$0 \longrightarrow E^{\otimes 2} \otimes L^\vee \longrightarrow E^{\otimes 2} \longrightarrow E^{\otimes 2}_{|C} \longrightarrow 0.$$

If we take cohomology, we see that since $h^1(S, E^{\otimes 2}) = 1$, $h^0(S, E^{\otimes 2}) = 3$, $h^0(C, E^{\otimes 2}_{|C}) = 3$, and $h^0(S, E^{\otimes 2} \otimes L^\vee) = 0$ (the latter as a consequence of $L.(E^{\otimes 2} \otimes L^\vee) = L.(c_1(M^\vee) \otimes L^\vee) < -\frac{1}{3}L^2$), we must have $h^1(S, E^{\otimes 2} \otimes L^\vee) \leq 1$. If $h^1(S, E^{\otimes 2} \otimes L^\vee) = 0$, then $h^0(C', E^{\otimes 2}_{|C'}) = 3$ for all $C' \in |L|_s$, and we are done. If $h^1(S, E^{\otimes 2} \otimes L^\vee) = 1$, then so is $h^1(S, L \otimes E^{\otimes(-2)})$, and by [9, Theorem], we either have $(L \otimes E^{\otimes(-2)})^2 = 0$ or there exists a (-2)-curve Γ such that $(L \otimes E^{\otimes(-2)}).\Gamma \leq -2$. From the condition $E^{\otimes 2}.L \leq \frac{1}{6}L^2$, we have $(L \otimes E^{\otimes(-2)})^2 \geq \frac{2}{3}L^2 > 0$, and so we are in the case where there exists a (-2)-curve Γ such that $(L \otimes E^{\otimes(-2)}).\Gamma \leq -2$.

We prove that $(L \otimes E^{\otimes(-2)}).\Gamma = -2$, thus contradicting the conditions of the theorem. If $(L \otimes E^{\otimes(-2)}).\Gamma < -2$, then $\Gamma^{\otimes 2}$ must be a base component of $L \otimes E^{\otimes(-2)}$. However, since $h^0(S, L \otimes E^{\otimes(-2)}) = h^0(S, L \otimes E^{\otimes(-2)} \otimes \Gamma^{\otimes(-2)})$, then $\chi(S, L \otimes E^{\otimes(-2)} \otimes \Gamma^{\otimes(-2)}) - \chi(S, L \otimes E^{\otimes(-2)}) \leq h^1(S, L \otimes E^{\otimes(-2)}) = 1$ (noting that no h^2 terms are positive because of the condition $E^{\otimes 2}.L \leq \frac{1}{6}L^2$); and Riemann-Roch gives us $-2(L \otimes E^{\otimes(-2)}).\Gamma - 4 \leq 1$, which is impossible.

Proof of part (c):We have $\mathrm{Cliff}(A) = d - 4 = c_2(\mathscr{F}_{C,A}) - 4$. We must prove that $\mathrm{Cliff}(D_{|C})$ is at most equal to this.

By definition, $\mathrm{Cliff}(D_{|C}) = D.L - 2(h^0(C, D_{|C}) - 1)$. Since $D = c_1(M^\vee)$, $D.L = c_1(M^\vee).c_1(N^\vee) + c_1(M^\vee)^2$ and $h^0(C, D_{|C}) \geq h^0(S, D) \geq \frac{1}{2}c_1(M^\vee)^2 + 2$, this immediately gives us

$$\mathrm{Cliff}(D_{|C}) \leq c_1(M^\vee).N^\vee - 2 = c_2(\mathscr{F}_{C,A}) - c_2(\tilde{M}) - 2,$$

where we recall that $\tilde{M}$ is as given in (2). We must prove that $c_2(\tilde{M}) \geq 2$.

Since $\tilde{M}$ is semistable, it follows from [7, Theorem 3.4.1] that $c_2(\tilde{M}) \geq \frac{1}{4}c_1(\tilde{M})^2$. In part (b) of the proof, we showed that $c_1(\tilde{M})^2 > 0$ (since $D = c_1(\tilde{M})$) except for the case when $D_{|C} = A$. Since the result follows trivially in this latter case, we can assume that $c_1(\tilde{M})^2 > 0$, and so $c_2(\tilde{M}) \geq 1$. In the following, we will suppose $c_2(\tilde{M}) = 1$ (and hence $c_1(\tilde{M})^2 \leq 4$) and show that this yields a contradiction.

First note that, by taking cohomology of (2) and recalling that $h^1(S, \mathscr{F}_{C,A}^\vee) = h^2(S, \mathscr{F}_{C,A}^\vee) = 0$, we see that $h^1(S, \tilde{M}) = h^2(S, \tilde{M}) = 0$ (we proved that $h^2(S, N^\vee) = h^0(S, N) = 0$ in part (a)). Also, since $\tilde{M}$ is of rank 2 and globally generated away from a finite set, it must sit inside an exact sequence

$$0 \longrightarrow R_1 \otimes \mathscr{I}_\nu \longrightarrow \tilde{M} \longrightarrow R_2 \otimes \mathscr{I}_\eta \longrightarrow 0, \tag{4}$$

where R_i are line-bundles and ν and η finite subschemes. We can furthermore assume that R_1 is effective since $\tilde{M}$ has global sections, and R_2 is globally generated and $R_2 \otimes \mathscr{I}_\eta$ globally generated away from a finite set; and hence $R_1.R_2 \geq 0$, $R_2^2 \geq 0$, and length(ν), length(η) ≤ 1. Note that $R_1.R_2 = 1 - $ length(η) $-$ length(ν). Also, [9, Theorem] gives us that $h^1(S, c_1(\tilde{M})) = 0$.

Case: length(η) $= 1$. In this case, $R_1.R_2 = $ length(ν) $= 0$, and since at least one R_i must satisfy $R_i^2 > 0$, Proposition 2.3 yields that $R_1^2 R_2^2 \leq 0$, and so either $R_1^2 < 0$, or $R_1^2 = 0$. (If $R_1^2 > 0$ with $R_2^2 = 0$, we get $R_2 = \mathscr{O}_S$, and then $R_2 \otimes \mathscr{I}_\eta$ has no global sections.) If $R_1^2 < 0$, then also $R_1.c_1(\tilde{M}) = R_1.(R_1 \otimes R_2) < 0$, and so R_1 is a base component of $c_1(\tilde{M})$. However, since $\tilde{M}$ is globally generated away from a finite set, then so must $c_1(\tilde{M})$, and we get a contradiction.

If $R_1^2 = 0$, then $(R_1 \otimes R_2)^2 = R_2^2$, and putting $\mathscr{O}_S(D_1) = R_2$ and $\mathscr{O}_S(D_2) = R_1 \otimes R_2$, we get equality in Proposition 2.3, and so $R_1 = \mathscr{O}_S$ (since numerical and linear equivalence is the same for line bundles on K3 surfaces). However, in that case, taking cohomology of (4) gives us that $h^1(S, R_2 \otimes \mathscr{I}_\eta) = 1$, while cohomology of the sequence

$$0 \longrightarrow R_2 \otimes \mathscr{I}_\eta \longrightarrow R_2 \longrightarrow \mathscr{O}_\eta \longrightarrow 0$$

yields $h^1(S, R_2 \otimes \mathscr{I}_\eta) = $ length(η) $- h^0(S, R_2) + h^0(S, R_2 \otimes \mathscr{I}_\eta) + h^1(S, R_2)$. Since $h^1(S, R_2) = 0$ by [9, Theorem] and $h^0(S, R_2 \otimes \mathscr{I}_\eta) = h^0(S, R_2) - 1$ since R_2 is globally generated, this gives us $h^1(S, R_2 \otimes \mathscr{I}_\eta) = $ length(η) $- 1 = 0$, a contradiction.

Case: length(ν) $= 1$. This case is similar to the previous case. Here we also have $R_1.R_2 = 0$, and in addition, length(η) $= 0$. Here, we cannot have

$R_1^2 > 0$ with $R_2^2 = 0$, because we get $R_2 = \mathcal{O}_S$, and dualising (4) would imply that $\tilde{M}^\vee = M$ has global sections, a contradiction. So the two alternatives are $R_1^2 < 0$ or $R_1^2 = 0$, as in the previous case. We cannot have $R_1^2 < 0$ for the same reason as in the previous case. If $R_1^2 = 0$ with $R_2^2 > 0$, we get $R_1 = \mathcal{O}_S$ as in the previous case, and $h^1(S, R_2) = 1$. However, since R_2 is globally generated with positive self-intersection, this is impossible by [9, Theorem].

Case: length$(\nu) = $ length$(\eta) = 0$. In this case, $R_1.R_2 = 1$, and since self intersection on a K3 surface is always even, Proposition 2.3 yields that $R_1^2 R_2^2 \leq 0$. If $R_1^2 < 0$, then it must be ≤ -2, and we get $R_1.(R_1 \otimes R_2) \leq -1$, and so R_1 is a base component of $c_1(\tilde{M})$, which contradicts $c_1(\tilde{M})$ being globally generated. It follows that $R_1^2 \geq 0$.

Having $R_1^2, R_2^2 \geq 0$ implies that $(R_2 \otimes R_1^\vee)^2 \geq -2$, and it follows from Riemann-Roch that either $R_2 \geq R_1$ or $R_1 \geq R_2$. By semistability of $\tilde{M}$, we have $R_1.L \leq \frac{1}{2}c_1(\tilde{M}).L = \frac{1}{2}(R_1 \otimes R_2).L$. If $R_1 \gneq R_2$, this would give us $\frac{1}{2}(R_1 \otimes R_2).L \lneq R_1.L$ (by ampleness of L), which is impossible. It follows that $R_2 \geq R_1$.

Now, since R_2 is globally generated, $(R_2 \otimes R_1^\vee).R_2 \geq 0$, and so $R_2^2 \geq 2$ and $c_1(\tilde{M})^2 \geq 4$. Since we originally had $c_1(\tilde{M})^2 \leq 4$ (as a consequence of assuming $c_2(\tilde{M}) = 1$), equality follows, together with $R_2^2 = 2$ and $R_1^2 = 0$. By [9, Theorem], $h^1(S, R_2) = 0$, and since $h^2(S, \tilde{M}) = 0$, we get $h^2(S, R_1) = 0$, and so $R_1 = E^{\otimes n}$ where E is an elliptic pencil. Since $R_1.R_2 = 1$, then $n = 1$.

Note that since $R_2 \otimes R_1^\vee > 0$ and $\tilde{M}$ is semistable, then $\tilde{M}$ must be a non-split extension of R_1 and R_2. As we saw above, the dimension of isomorphism classes of non-trivial extensions is $\mathrm{Ext}^1_{\mathcal{O}_S}(R_2, R_1) - 1 = h^1(S, R_2 \otimes R_1^\vee) - 1$, and so $h^1(S, R_2 \otimes R_1^\vee) > 0$. By [9, Theorem], this implies that either $R_2 \cong E \otimes E'$ where E' is an elliptic pencil satisfying $E'.E = 1$; or $(R_2 \otimes R_1^\vee).\Gamma \leq -2$ for some (-2)-curve Γ, implying that $R_2 = E \otimes B \otimes \Gamma^{\otimes m}$, where $m \geq 1$ is an integer and $B \geq 0$ is a possibly trivial line bundle satisfying $B.\Gamma \geq 0$.

If $R_2 = E \otimes E'$, then $h^1(S, R_2 \otimes R_1^\vee) = 0$, a contradiction.

If $R_2 = E \otimes B \otimes \Gamma^{\otimes m}$, note that since $R_1 = E$, then $1 = R_1.R_2 = E.B + mE.\Gamma \geq mE.\Gamma$. Since $(R_2 \otimes R_1).\Gamma \geq 0$ (recall that $R_1 \otimes R_2$ is globally generated) and $(R_2 \otimes R_1^\vee).\Gamma \leq -2$, we get $R_1.\Gamma \geq 1$. However, since $R_2.\Gamma \geq 0$ (recall that R_2 is globally generated), this means that $(R_2 \otimes R_1).\Gamma \geq 1$ instead of ≥ 0, and we end up with $R_1.\Gamma \geq 2$. But then $R_2.R_1 \geq mE.\Gamma \geq 2$, a contradiction.

We conclude that $\mathrm{Cliff}(D_{|C}) \leq \mathrm{Cliff}(A)$.

PROOF OF THEOREM 1.4.   The proof of Theorem 1.4 uses exactly the same techniques as in the proof of Theorem 1.2. We include it here for the sake of completion.

The condition on C and A are that $\rho(g, 1, d) < 0$. In this case, it follows that $\mathscr{F}_{C,A}$ is non-simple, and hence non-stable.

Part (a) is proved using the same diagram as in (3), the only difference being that $\mathrm{rk}(M) = 1$, and that $M.L \geq -\frac{1}{2}L^2$ and $c_1(N).L \leq -\frac{1}{2}L^2$. The latter inequality implies that N has no global sections, and so ϕ is injective on global sections. It follows, from the arguments in the proof of Theorem 1.2, that each global section of A comes from a unique global section of $\mathscr{G}$, and that the map must be the restriction map to C.

We now prove part (b): following the proof of Theorem 1.2(b), it is clear that $h^0(S, D) \geq 2$. We have $h^0(S, L \otimes D^\vee) = h^0(S, c_1(N)^\vee)$. To prove that the latter is ≥ 2, is suffices by Riemann-Roch to show that $c_1(N)^2 \geq 0$. We have $c_1(N).(M \otimes c_1(N)) = c_1(N).L^\vee \geq \frac{1}{2}L^2 = \frac{1}{2}M^2 + M.c_1(N) + \frac{1}{2}c_1(N)^2$, and so $\frac{1}{2}c_1(N)^2 \geq \frac{1}{2}M^2$. Since $c_1(\tilde{M}) = M^\vee$ and $\tilde{M}$ is globally generated away from a finite set, then $M^\vee$ is globally generated, and so $M^2 = (M^\vee)^2 \geq 0$, and we can conclude that $h^0(S, L \otimes D^\vee) \geq 2$.

The argument that $h^0(C, D_{|C})$ is independent of the curve C in $|L|_s$ is similar to the argument in the proof of Theorem 1.2. We see that no (-2)-curve can intersect D negatively, and so $h^1(S, D)$ can be positive only if $D^2 = 0$. We see that $0 = M.c_1(\mathscr{G}) + c_2(\mathscr{G})$, and so $D^2 = M^2 \geq c_2(\mathscr{G})$. Thus, $D^2 = 0$ if and only if $\mathscr{G} = D$. As a consequence, $h^0(S, D) = 2$, and so we must have $D = E$ where E is an elliptic pencil. In that case, $h^1(S, D) = 0$, and we conclude that $h^0(C, D_{|C})$ is independent of the curve C in $|L|_s$.

To prove (c), we have $\mathrm{Cliff}(A) = d - 2 = c_2(\mathscr{F}_{C,A}) - 2$; and $\mathrm{Cliff}(D_{|C}) = D.L - 2(h^0(C, D_{|C}) - 1) \leq c_2(\mathscr{F}_{C,A}) - c_2(\tilde{M}) - 2 \leq c_2(\mathscr{F}_{C,A}) - 2$, and so $\mathrm{Cliff}(D_{|C}) \leq \mathrm{Cliff}(A)$, as desired.

3. An example of a linear system $|L|$ where $h^0(C, E_{|C}^{\otimes 2})$ depends on the curve C in $|L|_s$

In this section, we give an example of a case where $h^0(C, E_{|C}^{\otimes 2})$ depends on the curve C in $|L|_s$, thus showing that the condition in Theorem 1.2 is necessary. We first state the main result of this section before presenting a proposition and lemma to help us with the construction.

THEOREM 3.1. *There exists a smooth K3 surface with an ample linear system $|L|$ and elliptic pencil E such that $h^0(C, E_{|C}^{\otimes 2})$ depends on the curve C in $|L|_s$.*

In order to obtain this, we will need a smooth K3 surface containing a (-2)-curve Γ in addition to an elliptic pencil E such that $E.\Gamma = 2$. Furthermore, to ensure ampleness of L, which we will define later, we need all divisors of $|E|$ to be irreducible. We start by proving that such a K3 surface exists.

PROPOSITION 3.2. *There exists a smooth K3 surface S in $\mathbb{P}^5$ which contains a (-2)-curve Γ and an elliptic pencil E such that $E.\Gamma = 2$ and all divisors of $|E|$ are irreducible.*

PROOF. We will obtain the K3 surface S by intersecting three quadric hypersurfaces Q_1, Q_2 and Q_3 in $\mathbb{P}^5$ such that the intersection is smooth and irreducible, and where the hypersurfaces satisfy the following conditions:

- the quadric Q_1 contains a linear three-space V_1 and hence also a pencil of three-spaces V_2 cut out by all hyperplanes H containing V_1;
- the three-space V_1 intersects Q_2 in the sum of two planes; and
- the intersections of Q_3 with one of the planes in $V_1 \cap Q_2$, and with $V_2 \cap Q_2$ for all V_2, are smooth and irreducible.

We begin the proof by arguing that the above conditions will indeed produce line bundles with the desired properties. First of all, the intersection of Q_3 with one of the planes in $V_1 \cap Q_2$ will be a plane conic, hence rational, and will be our desired (-2)-curve Γ. The pencil of three-spaces V_2 will give us the desired elliptic pencil E, where clearly all divisors of $|E|$ will be irreducible. The curves in $|E|$ are elliptic because they are complete intersections of two quadrics in $\mathbb{P}^3$, and the adjunction formula then gives us that the canonical bundle is trivial. We compute the intersection $E.\Gamma$ using the equality $(E + \Gamma)^2 = (H - \Gamma_2)^2$, where Γ_2 is the other plane conic in $V_1 \cap Q_2 \cap Q_3$. We have $(E + \Gamma)^2 = 2E.\Gamma - 2$, and $(H - \Gamma_2)^2 = 2$, using that $H.\Gamma_2 = 2$. It follows that $E.\Gamma = 2$.

We now construct Q_1, Q_2 and Q_3 with the above-listed properties. The idea of the construction is to build cones over quadric surfaces Z_i contained in linear three-spaces J_i, with certain lines T_i as base loci.

We start with Q_1. Let J_1 be any linear three-space, and let T_1 be a line that avoids J_1. Consider the linear system of quadric surfaces Z_1 in J_1. For given Z_1, we let Q_1 be the span of planes connecting T_1 with points on Z_1. This quadric fourfold is smooth outside of T_1 because of the following: by blowing up in T_1 and considering the strict transform $\tilde{Q}_1$ of Q_1, we see that by varying Z_1, the linear system we obtain in $\mathbb{P}^5 \times \mathbb{P}^3$ is base-point free, since the pre-image of T_1 corresponds to planes connecting T_1 with points in J_1. Thus, the general $\tilde{Q}_1$ is smooth by Bertini's theorem, and so the general Q_1 is smooth outside of T_1. We choose the desired three-space V_1 to be the span of T_1 with any line on Z_1. The pencil of three-spaces V_2 is, as previously mentioned, cut out by considering hyperplanes H containing V_1.

Before constructing Q_2, we first choose a line T_3 not contained in Q_1 and which avoids V_1, and a linear three-space J_3 avoiding T_3 and chosen such that $Q_1 \cap J_3$ is smooth and irreducible (which can be done by considering general $\tilde{J}_3$ on $\tilde{Q}_1$). These will be used to construct Q_3 later. Note that T_3 intersects Q_1 in two points.

We now construct Q_2: first, let T_2 be a line intersecting V_1 in a point; avoids T_1, T_3 and the two V_2's that intersect T_3; and is not contained in Q_1. We next consider all possible three-spaces J_2 and note that all intersect V_1 in at least a line ℓ. For each J_2 and each line ℓ in $J_2 \cap V_1$, consider the linear system of quadric surfaces Z_2 containing ℓ. The general J_2 will intersect V_1 in only a single line ℓ, and will avoid T_2 and T_1. For given J_2 and Z_2, we let Q_2 be given by the span of planes connecting T_2 with points on Z_2. The three-space V_1 then intersects Q_2 in two planes, where one is given by the span of $V_1 \cap T_2$ and the line ℓ. The general Q_2 intersects J_3 in a smooth irreducible surface, since the various Q_2 define a base-point free linear system on J_3 and we can thus apply Bertini's theorem.

It now remains to construct Q_3. We have already chosen T_3 and J_3. Consider the linear system of quadric surfaces Z_3 in J_3, and note that the general Z_3 intersects Q_2 in a smooth irreducible curve. We now construct Q_3 just as with Q_1 and Q_2 and connect planes between T_3 and points in Z_3. The resulting linear system of Q_3's cut out a base-point free linear system on the two planes in $V_1 \cap Q_2$. (Proof: let Π be one of the planes. Since T_3 avoids V_1, it in particular avoids Π. Each point on Π and T_3 span a plane, which intersects J_3 in a point. We can then choose Z_3 so that it avoids that point.) By Bertini's theorem, the general Q_3 cuts out a smooth conic on each of the planes in $V_1 \cap Q_2$.

To show that every element $V_2 \cap Q_2 \cap Q_3$ of the elliptic pencil $|E|$ is irreducible, first consider the V_2's that avoid T_3. It is then clear that $V_2 \cap Q_2 \cap Q_3$ is found by first intersecting Z_3 with Q_2, yielding a curve C in Z_3, and then considering the projection map to V_2 by planes connecting points on C with T_3. This projection map is clearly an isomorphism, and so as long as C is irreducible, then the corresponding curve in V_2 is also irreducible. In the two cases when V_2 intersects T_3, these three-spaces avoid T_2, by how T_2 was chosen, and so we can consider the curve $C' = Z_2 \cap Q_3$ and construct a similar projection map using planes through T_2. Since for general Q_2 and Q_3 we have $Z_2 \cap Q_3$ irreducible and $Z_3 \cap Q_2$ irreducible, we are done.

It remains to show that $Q_1 \cap Q_2 \cap Q_3$ is smooth. First of all, by blowing up in T_1, we have already seen that the general $\tilde{Q}_1$ is smooth. We now also blow up in T_2 and consider the linear system of $\tilde{\tilde{Q}}_2$'s on $\tilde{Q}_1$. Using the same argument as in the T_1 case (which is possible since the general J_2 avoids T_1), we see that the linear system of $\tilde{\tilde{Q}}_2$'s is also base-point free, and so the general element defines a smooth divisor on $\tilde{Q}_1$. We can choose Q_2 such that $\tilde{\tilde{Q}}_1 \cap \tilde{\tilde{Q}}_2$ avoids T_3, and so the linear system of $\tilde{\tilde{Q}}_3$'s restricted to $\tilde{\tilde{Q}}_1 \cap \tilde{\tilde{Q}}_2$ will then also be base-point free, and the general $\tilde{\tilde{Q}}_1 \cap \tilde{\tilde{Q}}_2 \cap \tilde{\tilde{Q}}_3$ is therefore smooth.

The general $Q_1 \cap Q_2 \cap Q_3$ avoids T_1, T_2 and T_3, and is thus isomorphic to $\tilde{Q}_1 \cap \tilde{Q}_2 \cap \tilde{Q}_3$. The result follows.

In the following, we will let S be a K3 surface as given in Proposition 3.2. We now consider the line-bundle $L = E^{\otimes a} \otimes \Gamma^{\otimes(a-1)}$ where $a \geq 3$ is an integer. (We let $a \geq 7$ if we wish the condition $\deg(A) \leq \frac{1}{6}L^2$ from Theorem 1.2 to be satisfied, but our example works for all $a \geq 3$.) In the following lemma, we prove that $|L|$ contains smooth irreducible curves, and that L is ample.

LEMMA 3.3. *Let L and S be as above. Then:*

(a) *the linear system $|L|$ contains smooth, irreducible curves; and*

(b) *the line-bundle L is ample.*

PROOF. In part (a), we prove that Γ is not a base component of $|L|$. Then, since linear systems on K3 surfaces without base components are base-point free, we can apply Bertini's theorem to conclude smoothness for the general curves. In order to prove that Γ is not a base component, we show that $h^0(S, L) > h^0(S, L \otimes \Gamma^\vee)$. Using [9, Theorem], $h^1(S, L) = h^1(S, L \otimes \Gamma^\vee) = 0$, and so $h^0(S, L) = \frac{1}{2}L^2 + 2$ and $h^0(S, L \otimes \Gamma^\vee) = \frac{1}{2}(L \otimes \Gamma^\vee)^2 + 2 = h^0(S, L) - L.\Gamma - 1 = h^0(S, L) - 3$. It follows that the general element of $|L|$ is smooth and irreducible.

To prove (b), we first of all have $L.E > 0$ and $L.\Gamma > 0$. It is also clear that $L.D \geq 0$ for any other irreducible curve D, since $|L|$ contains irreducible curves. Suppose $L.D = 0$ for some D. Since D can have neither E nor Γ as a component, it follows that $E.D = \Gamma.D = 0$. We now use Proposition 2.3 with $L = D_2$ and $D = D_1$, and see that $D^2 \leq 0$. Suppose first that $D^2 = 0$. Using Proposition 2.3 again, this time with $D_2 = L$ and $D_1 = L \otimes D^\vee$, it follows that D must be trivial.

Now suppose $D^2 < 0$. Since D is irreducible, this means that D is a (-2) curve. Consider the exact sequence

$$0 \longrightarrow E^\vee \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{E'} \longrightarrow 0,$$

where E' is a smooth elliptic curve in $|E|$. Tensor the sequence with D and take cohomology. We have $h^1(S, D) = 0$ and $h^1(E', D_{|E'}) = 1$, since $D_{|E'}$ is trivial (recall that $D.E = 0$) and $\omega_{E'} \cong \mathcal{O}_{E'}$. This gives us $h^2(S, D \otimes E^\vee) = h^0(S, E \otimes D^\vee) = 1$, implying that $D < E$, which contradicts Proposition 3.2. We can thus conclude that L is ample.

We now prove the main result of this section. The idea is to consider divisors in $|E^{\otimes 2} \otimes \Gamma|$ restricted to various curves C in $|L|_s$ and subtract $\Gamma \cap C$. For the general curves, we get the same linear system as from $|E^{\otimes 2}|$. However, we show that there exist other curves C' where divisors in $|E^{\otimes 2} \otimes \Gamma|$ without Γ

as a component intersect C' exactly in the points $\Gamma \cap C'$, and the linear system on C' thus obtains an extra dimension.

PROOF OF THEOREM 3.1.   Consider the exact sequence

$$0 \longrightarrow L^{\vee} \otimes E^{\otimes 2} \longrightarrow E^{\otimes 2} \longrightarrow E^{\otimes 2}_{|C} \longrightarrow 0, \tag{5}$$

where C is a curve in $|L|_s$. We argue that $h^1(S, L^{\vee} \otimes E^{\otimes 2}) = 1$. Note that this equals $h^1(S, L \otimes E^{\otimes(-2)})$, and by [9, Theorem], $h^1(S, L \otimes E^{\otimes(-2)}) > 0$ while $h^1(S, L \otimes E^{\otimes(-2)} \otimes \Gamma^{\vee}) = 0$. By comparing $\chi(S, L \otimes E^{\otimes(-2)})$ with $\chi(S, L \otimes E^{\otimes(-2)} \otimes \Gamma^{\vee})$, and using that $h^0(S, L \otimes E^{\otimes(-2)}) = h^0(S, L \otimes E^{\otimes(-2)} \otimes \Gamma^{\vee})$, we get that $h^1(S, L \otimes E^{\otimes(-2)}) = 1$.

Now tensor (5) with Γ and take cohomology. We see that $H^0(S, E^{\otimes 2} \otimes \Gamma) \cong H^0(C, E^{\otimes 2} \otimes \Gamma_{|C})$, and we can therefore conclude that the linear system $|E^{\otimes 2}_{|C}|$ is found precisely by considering divisors in $|E^{\otimes 2} \otimes \Gamma|$ that, restricted to C, are zero in $\Gamma \cap C$.

Note that divisors in $|E^{\otimes 2} \otimes \Gamma|$ that have Γ as a component will not cut out any extra divisors in $|E^{\otimes 2}_{|C}|$ apart from those already cut out by $|E^{\otimes 2}|$ on S. We must therefore consider curves in $|E^{\otimes 2} \otimes \Gamma|$ that do not have Γ as a component, but still cut through C exactly where C intersects Γ.

We now have two situations: first of all, consider curves J in $|E^{\otimes 2} \otimes \Gamma|$ that do not have Γ as a component. By considering the exact sequence

$$0 \longrightarrow E^{\otimes 2} \longrightarrow E^{\otimes 2} \otimes \Gamma \xrightarrow{\psi} (E^{\otimes 2} \otimes \Gamma)_{|\Gamma} \longrightarrow 0 \tag{6}$$

and noting that $h^1(S, E^{\otimes 2}) = 1$ while $h^1(S, E^{\otimes 2} \otimes \Gamma) = 0$, we see that by varying J, one dimension of divisors in $|(E^{\otimes 2} \otimes \Gamma)_{|\Gamma}| = |\mathcal{O}_{\Gamma}(2)|$ is cut out. The image of ψ cuts out a sub-linear system of $|\mathcal{O}_{\Gamma}(2)|$ that we denote by $\mathfrak{d}$. Since $|E^{\otimes 2} \otimes \Gamma|$ is base-component free and hence base-point free, it follows that $\mathfrak{d}$ is base-point free. Now consider the exact sequence

$$0 \longrightarrow L \otimes \Gamma^{\vee} \longrightarrow L \xrightarrow{\tau} L_{|\Gamma} \longrightarrow 0,$$

and note that $h^1(S, L \otimes \Gamma^{\vee}) = 0$, by [9, Theorem]. We see here that $|L|$ cuts out the entire linear system $|\mathcal{O}_{\Gamma}(2)|$, and we can then consider the pre-image of the sub-linear system $\mathfrak{d}$, which then is a sub-linear system $\mathfrak{L}$ of $|L|$ without base-points along Γ. We prove that $\mathfrak{L}$ is also base-point free everywhere else, so that we can apply Bertini's theorem and conclude that the general element of $\mathfrak{L}$ is smooth.

The proof is by induction on a in the expression $L = E^{\otimes a} \otimes \Gamma^{\otimes(a-1)}$, where we start with $a = 2$. In this case, the curves in L and in $|E^{\otimes 2} \otimes \Gamma|$ are the same, so the result is obvious. Now for each $a \geq 3$, suppose there

is a base-point free sub-linear system $\mathfrak{L}'$ of $|E^{\otimes(a-1)} \otimes \Gamma^{\otimes(a-2)}|$ consisting of divisors that intersect Γ where the curves J intersect. Consider the linear system $|E \otimes \Gamma|$. By Riemann-Roch, it contains divisors that do not have Γ as a component, and so it is clearly base-component free and hence base-point free. Also, since $(E \otimes \Gamma).\Gamma = 0$, the irreducible elements of this linear system avoids Γ. Denote these elements by V. It then follows that $\mathfrak{L}' + V$ is a subset of divisors in $|E^{\otimes a} \otimes \Gamma^{\otimes(a-1)}|$ that intersect Γ where J intersects. It is clearly base-point free, and so we can conclude that the sub-linear system $\mathfrak{L}$ is also base-point free. By Bertini's theorem, it follows that the general element of $\mathfrak{L}$ is smooth.

We conclude that for general $J \in |E^{\otimes 2} \otimes \Gamma|$ without Γ as a component, and smooth curves C' in $|L|$ that pass through $J \cap \Gamma$, there is an effective divisor $J \cap C' - \Gamma \cap C' \in |E^{\otimes 2}_{|C'}|$ which is not cut out by a divisor in $|E^{\otimes 2}|$. As a result, we get $h^0(C', E^{\otimes 2}_{|C'}) = 4$.

It remains to prove that there exists a smooth, irreducible curve $C'' \in |L|$ where the above situation does not occur. It then suffices to find a curve C'' such that no divisor J in $|E^{\otimes 2} \otimes \Gamma|$ satisfies $J \cap \Gamma = C'' \cap \Gamma$. This follows by the exact sequence (6), where we saw that ψ is not surjective on global sections and hence that the general divisors $Z \in |(E^{\otimes 2} \otimes \Gamma)_{|\Gamma}|$ are not cut out by any of the divisors in $|E^{\otimes 2} \otimes \Gamma|$. From the argument above, it follows that any curve C'' that cuts out such a divisor Z on Γ will satisfy $h^0(C'', E^{\otimes 2}_{|C''}) = 3$.

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